Non-reverse Motion of a Two-body System along a Straight Line on a Rough Horizontal Plane

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Summary. The motion of two interacting bodies along a straight line on a horizontal plane is studied. Coulomb's dry friction is assumed to act between the bodies and the supporting plane. The distance between the bodies and their velocities change periodically in time. Necessary and sufficient conditions for the existence of a motion in which neither of the bodies moves in the direction opposite to that of the displacement of the system for the period are found. Such motions are called non-reverse motions. The advantage of the non-reverse motions is that they provide a minimum for the energy consumption on friction per unit path.

Statement of the problem

Consider a system of two interacting bodies (point masses) moving along a straight line on a horizontal plane. Coulomb's dry friction acts between the bodies and the supporting plane. We do not impose any constraints on the force of interaction of the bodies, in particular, we admit instantaneous redistribution of the linear momenta of the bodies. The interaction between the bodies by means of internal forces causes a change in the velocities of these bodies, thus leading to a change in the force of friction, which are external forces for the system under consideration. Let m and M be the masses of the bodies. In what follows, we will refer to these bodies as body m and body M. Let x and y denote the coordinates of bodies m and M measured along the line of the motion relative to a fixed (inertial) reference frame; v and V the velocities of the respective bodies relative to the inertial reference frame; k_m and k_M the coefficients of friction of the bodies against the supporting plane; F the force applied by body m to body M; and g the acceleration due to gravity. The motion of the system is governed by the equations

$$\begin{aligned} &\dot{x} = v, \quad \dot{y} = V, \quad m\dot{v} = -F + F_1, \quad M\dot{V} = F + F_2, \\ &F_1 = -k_m \operatorname{sgn} v, \quad v \neq 0, \qquad |F_1| \leq k_m m g, \quad v = 0, \\ &F_2 = -k_M \operatorname{sgn} V, \quad V \neq 0, \qquad |F_2| \leq k_M M g, \quad V = 0. \end{aligned} \tag{1}$$

We will be interested in the motions in which the quantities $y - x$, v, and V are T-periodic functions of time. Accordingly, we can confine our consideration to the time interval $[0,\, T]$.
Introduce the dimensionless variables

$$\tilde{x} = \frac{x}{k_m g T^2}, \quad \tilde{y} = \frac{y}{k_m g T^2}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{v} = \frac{v}{k_m g T}, \quad \tilde{V} = \frac{V}{k_m g T}, \quad \mu = \frac{m}{M}, \quad k = \frac{k_M}{k_m}, \quad u = \frac{F}{k_m m g}. \tag{2}$$

Then equations (1) become (the dot stands for differentiation with respect to $\tilde{t}$, the tildes are omitted)

$$\begin{aligned} &\dot{x} = v, \quad \dot{y} = V, \quad \dot{v} = -u + f_1, \quad \dot{V} = \mu u + k f_2, \\ &f_1 = -\operatorname{sgn} v, \quad v \neq 0, \qquad |f_1| \leq 1, \quad v = 0, \\ &f_2 = -\operatorname{sgn} V, \quad V \neq 0, \qquad |f_2| \leq 1, \quad V = 0. \end{aligned} \tag{3}$$

The periodicity of the quantities $y - x$, v, and V implies the following boundary conditions:

$$y(0) - x(0) = y(1) - x(1), \tag{4}$$

$$v(0) = v(1), \quad V(0) = V(1). \tag{5}$$

Without loss of generality relation (4) can be replaced by the relations $y(0) = x(0)$ and $y(1) = x(1)$. If $y(0) - x(0) = l \neq 0$, we can shift the point from which the coordinate y is measured by l along the line of motion.

We will be interested in the issue of whether the motion of system (3) subject to the boundary conditions (4) and (5) and the additional conditions

$$v(t) \geq 0, \quad V(t) \geq 0, \quad t \in [0,\, 1] \tag{6}$$

is possible. These conditions imply that both bodies always move in the same direction. They can stop for some time but neither of the bodies is allowed to move backward. We will call the motions that satisfy inequalities (6) non-reverse motions. The non-reverse motions minimize the energy consumption on friction per unit path.

Other aspects of the motion of a two-body locomotion system similar to that considered in our paper have been investigated in [1 – 3], [5, 6]. Our paper continues these studies. A non-reverse optimal motion of a chain of n bodies ($n > 2$) has been constructed in [4].

Basic result

Let $\mu \neq k$ ($k_m m \neq k_M M$). If $\mu = k$, the system cannot start moving progressively from a state of rest. More precisely, if $\mu = k$, $v(0) = 0$, and $V(0) = 0$, then the center of mass of the system will remain in its initial position, irrespective of the control law $u(t)$. We will not consider this case. Then without loss of generality we assume $\mu > k$ ($k_m m > k_M M$).

Proposition. For $\mu > k$ ($k_m m > k_M M$), the non-reverse motion of the system governed by equations (3) subject to boundary conditions (4) and (5) exist if and only if $\mu < 1$ ($m < M$).

This proposition implies that the non-reverse motion of the system is possible if and only if the sliding friction force for the body of larger mass is less than the sliding friction force for the body of smaller mass.

We will outline the idea of the proof of this proposition. We seek the minimum and the maximum for the quantity $\Delta = y(1) - x(1)$. The extrema are calculated over the functions $x(t)$ and $y(t)$ that satisfy equations (3) subject to the conditions $x(0) = y(0)$, (5), and (6) for various control laws $u(t)$. Recall that we do not impose constraints on the control variable u. In particular, we admit an impulsive control represented by Dirac's delta function, allowing thereby the velocities of the bodies to have jump discontinuities. It is proved that the motions that provide a maximum and a minimum for the quantity Δ do not have time intervals in which $v(t) = 0$ and $V(t) = 0$ simultaneously. Then we seek a minimum and a maximum of the quantity Δ among the motions in which the bodies move alternately, coming to a stop instantaneously and transmitting their momenta to the other body. If $\min \Delta \leq 0$ and $\max \Delta \geq 0$, then the desired motion is possible.

Example

Consider a simple implementation for the non-reverse motion. Let $x(0) = y(0) = 0$, $v(0) = V(0) = 0$, and let the control law be defined by

$$F = \theta_1[H(t) - H(t - t_*)] - MV(t_*)\delta(t - t_*) + \theta_2 H(t - t_*), \quad t \in [0,\, T], \tag{7}$$

where θ_1, θ_2, and t_* are constant parameters of the control law; $H(\cdot)$ is the Heaviside step function. The parameters θ_1, θ_2, and t_* should be adjusted to provide the following behavior for the system: during the time interval $[0,\, t_*)$, body M is accelerating forward ($\dot{V} > 0$), while body m is not moving; at the instant t_*, body M transmits its momentum to body m and comes to a stop; during the time interval $(t_*,\, T)$, body m is moving forward ($v > 0$) and is decelerating ($\dot{v} < 0$), while body M is not moving. At the time instant T, the boundary conditions $x(T) = y(T)$ and $v(T) = V(T) = 0$ must hold. According to these requirements, the parameters of the control law must satisfy the relations

$$V(T) = 0, \quad x(T) = x(t_*), \quad Mk_M g < \theta_1 \leq mk_m g, \quad -Mk_M g \leq \theta_2 \leq Mk_M g. \tag{8}$$

The functions $x(t)$, $y(t)$, $v(t)$, and $V(t)$ are obtained by solving equations (1) subject to zero initial conditions. Conditions (8) imply the following relations:

$$\theta_2 = Mm^{-1}(\theta_1 - k_M M g) - k_m m g, \quad t_* = TM(M+m)^{-1}, \tag{9}$$

$$k_M M g + mM^{-1}(k_m m - k_M M)\, g \leq \theta_1 \leq \min\left[k_m m g,\, k_M M g + mM^{-1}(k_m m + k_M M)\, g\right]. \tag{10}$$

Since $k_m m - k_M M > 0$ and $m < M$, domain (10) is non-empty. Any parameters satisfying relations (9) and (10) provide the desired behavior for the system.

Conclusions

The motion of a two-body locomotion system along a straight line on a rough horizontal plane is studied. The possibilities for the motion in which neither of the bodies moves in the direction opposite to that of the displacement of the system for the period are investigated. Such motions provide a minimum for the energy consumption on friction per unit path. It is proved that the desired motion is possible if and only if the sliding friction force for the body of larger mass is less than the sliding friction force for the body of smaller mass. An example of such a motion is given.

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