Bifurcations of periodic solutions for systems with discontinuities

J. P. Meijaard*
*Olton Engineering Consultancy, Enschede, The Netherlands

Summary. Dynamical systems with discontinuities are considered. The systems are piecewise smooth with the phase space partitioned by hypersurfaces where the non-smoothness occurs. Transition conditions for the state, the first variation and the second variation at these surfaces are derived in a general form. Although the system contains discontinuities, the overall system evolution is smooth if the transition surfaces are transversally intersected by the solution curves. Periodic solutions and their smooth bifurcations are calculated by a shooting method together with an arc continuation method. The methods are illustrated by examples with impacts, friction and bilinear spring stiffness.

Extended abstract

Problem description

The evolution of the state of a nonlinear dynamical system is described by a vector differential equation,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{1}$$

where $\boldsymbol{x}$ represents the state in an n-dimensional phase space, t is the time and a dot over a variable denotes a derivative with respect to time. Systems with discontinuities are characterized by codimension-one hypersurfaces in the phase space on which the state, the right-hand sides of the differential equations or some derivative of the right-hand side may change by a jump. A jump in the state may occur in mechanical systems with impacts, where the velocities may change instantaneously, the right-hand side may have a jump if the force on a system changes abruptly or in systems with dry friction if the sliding velocity changes sign, and a derivative may change for instance for systems with a bilinear spring stiffness.

The evolution of a discontinuous system consists of the smooth evolution on each subregion of the phase space between hypersurfaces of discontinuity and transition conditions at these hypersurfaces. As long as the solution crosses the hypersurfaces transversally and the hypersurfaces and the transition conditions are smooth, the whole evolution consists of the stitching together of smooth parts and is therefore as a whole smooth, except at the hypersurfaces. This means that all well-known ordinary bifurcations of periodic solutions can occur in these non-smooth systems. In addition, special bifurcations that can only occur in non-smooth systems may be found if the condition of transversal intersection at some hypersurface is violated, which will not be investigated here. In this presentation, the calculation of periodic solutions of non-smooth systems, their continuation if a parameter is changed and the continuation of bifurcations of these periodic solutions if more parameters are allowed to vary will be discussed. In a sense, it is a generalization and a completion of results presented earlier [1].

Transition conditions

Let a hypersurface be defined by the equation

$$g(\boldsymbol{x}, t) = 0 \tag{2}$$

and if the solution crosses the hypersurface at the time t, let the transition condition for the state be given by

$$\boldsymbol{x}^+ = \boldsymbol{S}(\boldsymbol{x}^-, t), \tag{3}$$

where a superscript plus denotes a variable just after the transition and a superscript minus a variable just before the transition.

The variational equations describe the evolution of small perturbations of the state $\delta\boldsymbol{x}$, which may be thought of as derivatives with respect to the initial conditions or with respect to system parameters, as

$$\delta\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\delta\boldsymbol{x}. \tag{4}$$

The transition conditions for these variations can be evaluated as (see for instance [2])

$$\delta\boldsymbol{x}^+ = \boldsymbol{T}\delta\boldsymbol{x}^-, \qquad \boldsymbol{T} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{x}} - \frac{\left[\dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{x}}\boldsymbol{f}^- + \dfrac{\partial \boldsymbol{S}}{\partial t} - \boldsymbol{f}^+\right]\dfrac{\partial g}{\partial \boldsymbol{x}}}{\dfrac{\partial g}{\partial \boldsymbol{x}}\boldsymbol{f}^- + \dfrac{\partial g}{\partial t}}. \tag{5}$$

This can be simplified somewhat if $\boldsymbol{S}$ is the identity and the system is autonomous.

Periodic solutions together with their stability can be found by a shooting method and the evaluation of the monodromy matrix and a branch of periodic solutions can be found by an arc continuation method if a parameter of the system is made variable. The arc continuation method calculates a solution near some known solution in two steps: first, a prediction step is made by some numerical extrapolation and then an accurate solution is found by an iterative method.

For the continuation of bifurcations, a second parameter has to be made variable. As variations of eigenvalues of the monodromy matrix are needed, second-order variations with their transition conditions have to be determined. These can be found by taking another derivative of the variational equations (4) and the transition condition (5). In order to reduce the number of equations that have to be integrated, an adjoint variable method for the calculation of the variations of the eigenvalues of the monodromy matrix is used. As the transition condition for the adjoint variables contains the inverse of the matrix $\boldsymbol{T}$, the method breaks down if this matrix is singular, as can occur in systems with friction. This difficulty can be circumvented by realizing that the motion effectively takes places in a lower-dimensional subspace of the phase space in this case.
Higher-codimension bifurcations can be handled in a similar way if they are still characterized by conditions on the eigenvalues of the monodromy matrix.

Some illustrative applications

The methods will be illustrated for some example problems. The first is a system with impacts, then a system with dry friction is considered and finally a system with a bilinear stiffness element is considered.

Conclusions

General transition conditions for non-smooth systems have been derived. These have been applied in methods to calculate periodic solutions and their bifurcations. Although specific cases have been treated in the past, this general framework has not been presented in the open literature so far.

References

[1] Meijaard, J.P (1991) Dynamics of Mechanical Systems, Algorithms for a Numerical Investigation of the Behaviour of Non-Linear Discrete Models. *Doctoral Thesis*, Delft University of Technology, Delft.

[2] Müller, P.C. (1995) Calculation of Lyapunov Exponents for Dynamic Systems with Discontinuities. *Chaos, Solitons & Fractals* **5**:1671-1681.