Approximation of Top Lyapunov Exponent of Stochastic Delayed Turning Model Using Fokker-Planck Approach

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Summary. In this work an effective method is presented to approximate the top Lyapunov exponent of stochastic delayed differential equations, through an example. Namely, the inhomogeneous material properties are modeled in regenerative turning processes by introducing white noise in the cutting coefficient. The model is a one degree of freedom linear delayed oscillator with stochastic parameters. The method is used to determine the stability of the process using the approximated top Lyapunov exponent. The results are compared with numerical simulations.

The Stochastic Model of Turning

In the field of manufacturing science machine tool vibration is a source of many problems. The commonly used models involve deterministic delay differential equations, in which the parameters are usually considered to be constant. During the measurements of these parameters, the average is considered and the variance is attributed to the quality of the measurement. However it is easy to see that materials are not homogeneous, so variance can be an inherent property, which leads us to the use of stochastic models.
The dimensionless regenerative turning model [1, 2, 3] is extended by a white noise term to describe the stochastic property of the cutting coefficient. To model this behavior the following linear dimensionless cutting force is model is used:

$$F_c = (\kappa + \sigma\Gamma_t)\, h_t, \tag{1}$$
$$h_t = 1 - \tilde{x}_t + \tilde{x}_{t-\tau}, \tag{2}$$

where h_t is the dimensionless chip thickness, $\tilde{x}_t$ is the position of the tool, ζ is the damping coefficient, τ is the time delay, κ is the mean of the dimensionless cutting force coefficient, σ is the noise intensity generated by the cutting force, and Γ_t is the Langevin force, which is used to model a standard Gaussian white noise. The differential equation of the dimensionless linear oscillator driven by the cutting force in (1):

$$\ddot{\tilde{x}}_t + 2\zeta\dot{\tilde{x}}_t + \tilde{x}_t = (\kappa + \sigma\Gamma_t)\,(1 - \tilde{x}_t + \tilde{x}_{t-\tau}). \tag{3}$$

Introducing a perturbed stationary solution:

$$\tilde{x}_t = x_{t,st} + x_t. \tag{4}$$

In the case of the existence of the stationary solution $x_{t,st}$ the increment form of the corresponding perturbed dimensionless governing equation is given by

$$\mathrm{d}x_t = y_t \mathrm{d}t, \tag{5}$$
$$\mathrm{d}y_t = -\left(2\zeta y_t + (1+\kappa)\, x_t - \kappa x_{t-\tau}\right) \mathrm{d}t + \sigma\,(x_{t-\tau} - x_t)\, \mathrm{d}W_t, \tag{6}$$

where x_t is the perturbed displacement, the y_t is the perturbed velocity of the cutting tool and $\mathrm{d}W_t$ is the Wiener increment with the properties $\mathbb{E}\,(\mathrm{d}W_t) = 0$, $\mathbb{E}\left(\mathrm{d}W_t^2\right) = \mathrm{d}t$, $\mathbb{E}\,(\mathrm{d}W_t \mathrm{d}W_s) = 0$ for $s \neq t$. To investigate the top Lyapunov exponent of the stationary solution the polar variables are introduced:

$$\phi_t = \arctan\left(\frac{y_t}{x_t}\right), \qquad A_t = \sqrt{x_t^2 + y_t^2}. \tag{7}$$

Using the Itô Lemma [4] the quantities (7) can be written in the form:

$$\mathrm{d}\phi_t = f_\phi\,(\phi_t, \phi_{t-\tau}, r_A(\tau))\, \mathrm{d}t + g_\phi\,(\phi_t, \phi_{t-\tau}, r_A(\tau))\, \mathrm{d}W_t, \tag{8}$$
$$\frac{\mathrm{d}A_t}{A_t} = f_A\,(\phi_t, \phi_{t-\tau}, r_A(\tau))\, \mathrm{d}t + g_A\,(\phi_t, \phi_{t-\tau}, r_A(\tau))\, \mathrm{d}W_t, \tag{9}$$
$$r_A(\tau) := \frac{A_{t-\tau}}{A_t} \tag{10}$$

For the sake of simplicity the notation of the variables of nonlinear functions $f_A,\ g_A,\ f_\phi,\ g_\phi$ will be omitted in the following, and the functions are detailed in the Appendix A. Using the Itô Lemma we can introduce the increment equation of the logarithm of the amplitude process as

$$\mathrm{d}\ln(A_t) = \left(f_A - \frac{1}{2}g_A^2\right) \mathrm{d}t + g_A \mathrm{d}W_t. \tag{11}$$

Provided that the phase angle process in (8) is stationary, the stationary phase velocity and the top Lyapunov exponent (TLE) for the enveloping exponential process can be calculated with the help of the expectation value of the equation (8) and (11), respectively [5]. For a linear delay differential equation there are infinite characteristic exponents, but in the stationary case only the effect of the one with the largest real value survives. The real value of this largest characteristic exponent is denoted with δ_{top} and the corresponding phase angle velocity, the imaginary part is the α_{top}:

$$\mathbb{E}\left(\mathrm{d}\phi\right) = \mathbb{E}\left(f_\phi\right)\mathrm{d}t = \alpha_{top}\mathrm{d}t, \tag{12}$$

$$\mathbb{E}\left(\mathrm{d}\ln A_t\right) = \mathbb{E}\left(f_A - \frac{1}{2}g_A^2\right)\mathrm{d}t = \delta_{top}\mathrm{d}t. \tag{13}$$

To calculate these expectation values we need the stationary probability density function (sPDF) of the variables ϕ_t and $r_A(\tau)$. It can be determined by Monte-Carlo simulations using e.g. the Euler-Maruyama method, which is usually very time consuming, because many realizations are needed with long simulated time. An other method is to write the corresponding Fokker-Planck equation and solve it, but for delayed stochastic differential equation it is possible only for a few cases [6].
A further possibility to calculate the TLE, if the following average is considered [5]:

$$\delta_{top} = \lim_{t\to\infty}\frac{1}{t}\ln(A_t) \quad \text{(a.s.)}, \tag{14}$$

$$\alpha_{top} = \lim_{t\to\infty}\frac{1}{t}\phi_t \quad \text{(a.s.)}. \tag{15}$$

These quantities can be calculated using the stochastic increment equations (8) and (9).

Approximation of the TLE using the Fokker Planck equation

To get the stationary Fokker Planck equation, only the above mentioned, separated dominant mode is considered. This approach simplifies the delayed stochastic differential equations to ordinary stochastic differential equations. The following approximations are applied according to these consideration:

$$\mathbb{E}\left(r_A(\tau)\right) = e^{-\delta_{top}\tau}, \qquad \mathbb{E}\left(\phi_{t-\tau}\right) = \phi_t - \alpha_{top}\tau. \tag{16}$$

To further simplify, the quantities δ_{top} and α_{top} in (16) are considered as independent from everything else and constant. Substituting (16) into (8)-(9) we get two stochastic differential equations, which depend only on the phase angle ϕ_t:

$$f_\phi\left(\phi_t, \phi_{t-\tau}, r_A(\tau)\right) \;\to\; f_\phi\left(\phi_t\right), \qquad g_\phi\left(\phi_t, \phi_{t-\tau}, r_A(\tau)\right) \;\to\; g_\phi\left(\phi_t\right) \tag{17}$$

$$f_A\left(\phi_t, \phi_{t-\tau}, r_A(\tau)\right) \;\to\; f_A\left(\phi_t\right), \qquad g_A\left(\phi_t, \phi_{t-\tau}, r_A(\tau)\right) \;\to\; g_A\left(\phi_t\right) \tag{18}$$

Substituting the approximations in (16)-(18) into (8) the stationary Fokker-Planck equation for the sPDE of the phase angle process can be written:

$$\left(g_\phi(\phi)\frac{\partial g_\phi(\phi)}{\partial\phi} - f_\phi(\phi)\right)p(\phi) + \frac{1}{2}g_\phi^2(\phi)\frac{\partial p(\phi)}{\partial\phi} = p_0, \tag{19}$$

where p_0 is an integration constant. The solution function for the sPDF $p(\phi) = \mathbb{P}(\phi_t \in [\phi, \phi + \mathrm{d}\phi])$ is a π periodic function and can be determined semi-analytically with Fourier series or numerically by means of a backwards Euler method (detailed in Appendix B) [5]. We choose p_0 to normalize the sPDF to 1 on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} p(\phi)\mathrm{d}\phi = 1. \tag{20}$$

Having the $p\left(\phi\right)$ sPDE the expectated values of TLE and the corresponding phase angle velocity can be calculated:

$$\delta_{top} = \int_{-\pi}^{\pi}\left(f_A(\phi) - \frac{1}{2}g_A(\phi)^2\right)p(\phi)\mathrm{d}\phi, \qquad \alpha_{top} = \int_{-\pi}^{\pi} f_\phi(\phi)p(\phi)\mathrm{d}\phi. \tag{21}$$

A difficulty of this approach is that the sPDF - through the Fokker-Planck equation (19) - contains implicitly the calculated quantities. Although the solution can be determined e.g. via an easily implementable iterative method where an initial value has to be chosen for δ_{top} and α_{top} to calculate a new value, then continuing the iteration until it converges. Normally this calculation does not converge, so the following relaxation has to be used:

$$\delta_{top,\text{new}} = k_\delta * \delta_{top,\text{old}} + (1 - k_\delta) * \delta_{top,\text{calculated}} \tag{22}$$

$$\alpha_{top,\text{new}} = k_\alpha * \alpha_{top,\text{old}} + (1 - k_\alpha) * \alpha_{top,\text{calculated}} \tag{23}$$

Since the solution of delayed equation (5)-(6) consists of infinite characteristic exponents, the converged δ_{top} and α_{top} strongly depends on the initial values for the iteration. This can be overcome by using a bisection method [7] to calculate all the TLE-s and phase angle velocities for a given parameter set, and then taking the maximum TLE and the phase corresponding velocity.

Numerical validation

To numerically validate the results, two main methods are used. Both requires the use of Monte-Carlo simulation of the realizations. The first method uses the following assumption for the stationary amplitude and phase angle evolution, respectively:

$$A_{t+\Delta t} = A_t e^{\delta \Delta t}, \qquad \phi_{t+\Delta t} = \phi_t + \alpha \Delta t. \tag{24}$$

From these the numerical approximations for the average TLE δ and the average phase angle velocity α c2an be expressed:

$$\delta = \frac{1}{T} \sum_{i=i_{\text{first}}}^{N-1} \ln\left(\frac{A_{i+1}}{A_i}\right), \qquad \alpha = \frac{1}{T} \sum_{i=i_{\text{first}}}^{N-1} \left(\phi_{i+1} - \phi_i\right), \tag{25}$$

where $T = (N - i_{\text{first}})\,\Delta t$ is the length of the investigated time period, i_{first} is the first simulation point, where the process can be assumed to be stationary, N is the number of simulated time point, Δt is the time step of the simulation and $A_i = A_{i \cdot \Delta t}$ and $\phi_i = \phi_{i \cdot \Delta t}$ are the amplitude and phase angle processes at time $i\Delta t$, respectively.
The other method is based on the stationarity and ergodicity of the process: the sPDF can be calculated with the use of the distribution of the stationary trajectories along time. Using the sPDF calculated by the trajectories the integrands in (21) can be evaluated numerically.

Results

Iteration method

On Fig. 1 the calculated TLE can be seen for a fixed delay τ along κ. The different color lines denote the different methods used to calculate the TLE: ■ (yellow) - TLE of the realizations calculated by the exponential envelope of the trajectories, detailed in (25), ■ (red) - substituting the realizations into (12) and (13), ■ (green) and ■ (blue) - proposed iterative Fokker-Planck method using different initial values. It can be observed, that the proposed method gives an upper approximation for the TLE in this test case. Furthermore the computation is two order of magnitude faster and gives a smooth function in κ, but it is very sensitive for the initial values of the iteration to get δ_{top} and α_{top}. On Fig. 2 the sPDF of the variable ϕ_t is shown at the parameters marked with the dashed line on Fig. 1. The ■ (yellow) bins show the sPDF gained by Monte-Carlo simulations and the ■ (blue) and ■ (green) lines show the sPDF calculated with the approximated Fokker-Planck equation (the colors are matched with the ones used on Fig. 1). It can be seen that the semi-analytic Fokker-Planck approach gives a sufficient approximation for the sPDF if calculated with the correct TLE, if the correct δ_{top} - α_{top} is found.

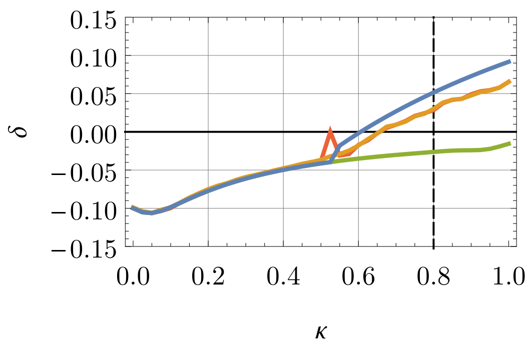


Figure 1: An example for the values of the TLE at $^{2\pi}/_{\tau} = 0.95$ using different methods

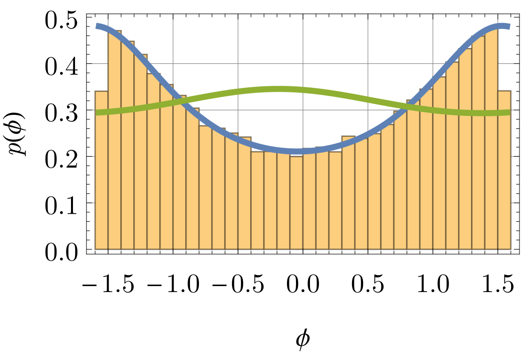


Figure 2: An example for the sPDF of the phase angle at $^{2\pi}/_{\tau} = 0.95$ and $\kappa = 0.8$ (dashed line on Fig. 1)

Multi dimensional bisection method

Since the results of the calculation of the TLE with the iteration method strongly depend on the initial values, it cannot be well automatized (e.g. for stability map calculation). A multi dimensional bisection method can be used for this purpose [7]. The solutions for the TLE δ_{top} and phase angle velocity α_{top} along the dimensionless κ parameters create multiple spatial curves in the κ - δ_{top} - α_{top} basis, which can be determined with the method.
A few example for the resulting curves obtained with the combination of the multi dimensional bisection method combined with the described Fokker-Planck approximation can be seen on Fig. 3. The results obtained with different method are shown with different colors: ■ (blue) shows the solutions with the combination of the proposed Fokker-Planck method and multi-dimensional bisection method, ■ (yellow) denotes the calculated TLE and phase angle velocity of the trajectories, and the ■ (red) is gained by substituting the realizations into (12) and (13). On the shown sections, it can be seen, that the method gives a reliable approximation, even if there are multiple solutions present for the δ_{top}.

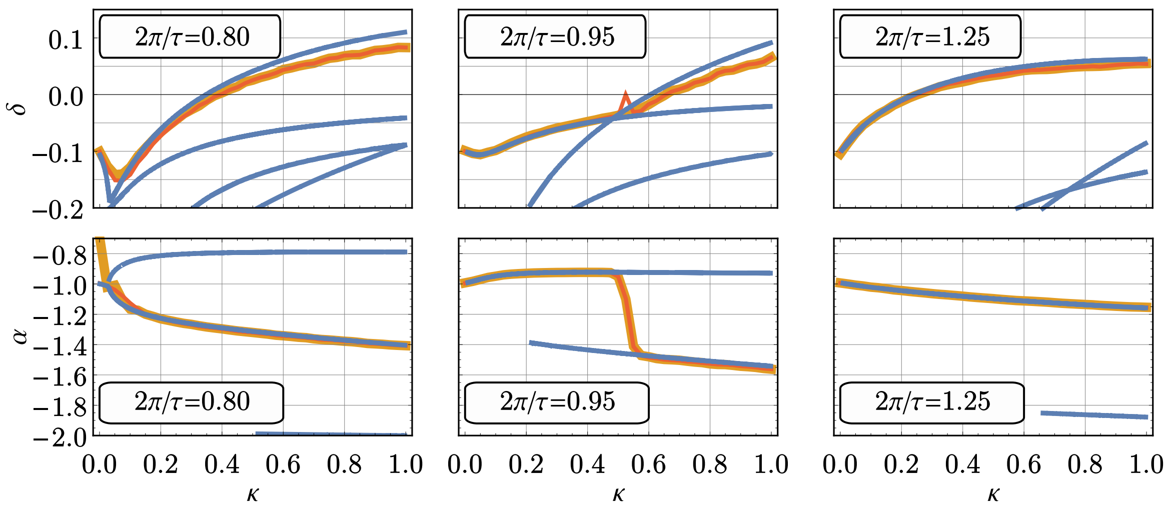
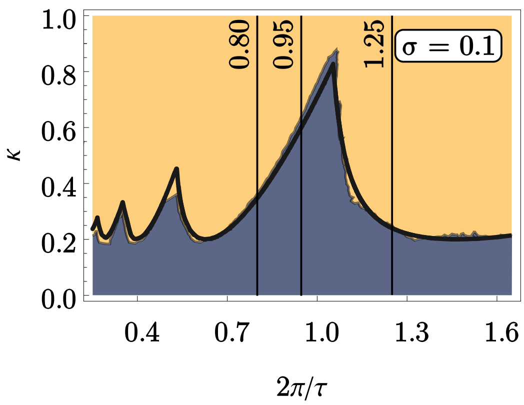


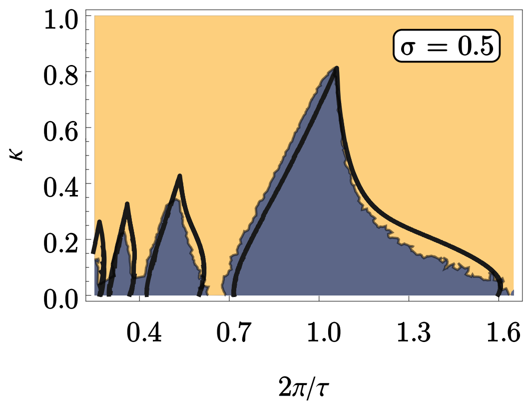
Figure 3: TLEs and the corresponding phase angle velocities, compared with numerical results at $\sigma = 0.1$

Stability maps

Taking the maximum of the calculated δ_{top} values, a stability chart can be determined. If this $\delta_{top} < 0$, then the stationary solution is stable, and if the $\delta_{top} > 0$ then it is unstable. On Fig. 4 the resulting stability borders (■ (black) curves) can be seen for $\sigma = 0.1$ and $\sigma = 0.5$, along with the numerically determined stable ■ (blue) and unstable ■ (yellow) areas. On Fig. 4a the parameter lines on which the results on Fig. 3 where calculated are marked. It can be seen, that the presence of the noise destabilizes this system. On Fig. 4b it can be seen, that this destabilization effect mainly occur at the bottom of the lobes. This is due to the fact that the σ intensity of the fluctuation is constant along the map: the noise has larger effect on the behavior for smaller κ values, since the relative variance of the cutting force is larger, and it can also became negative. This causes the bottom of the lobes to move downward and even vanish but the peaks of the lobes move little to nothing.



(a) Stability chart of the system for $\sigma = 0.1$



(b) Stability chart of the system for $\sigma = 0.5$

Figure 4: Stability maps of the stationary solution of the presented stochastic turning model

Conclusion

Parameter fluctuations due to material inhomogenity during turning is taken into account as a white noise process in the cutting coefficient. With increasing intensity it destabilizes the stationary solution of the system, but in this particular system this occurs only at unproportionally large σ values.
The proposed method gives a good approximation for the stability maps for higher σ values too, where the numerical methods are not so reliable. As it can be seen on Fig. 4b, the proposed semi-analytic method not necessarily gives an upper estimation of the TLE for larger σ values, but difference between the stability borders calculated with the different methods can also be due to the destabilizing effect of the Euler method.
The main advantage of the method is in the computational time. A stability chart calculation based on Monte-Carlo simulation takes $\sim$ 8 - 9000 seconds, and are unreliable near the stability borders (see Fig. 4), and can even contain the effects of the used numerical integration method. However, the chart calculation with proposed method only takes $\sim$ 50 - 60 seconds, which is 2 order of magnitude smaller. Furthermore the method gives a smoother stability border, which is expected, in contrast of the borders gained with the use of other numerical methods.

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Appendix A - Components of the increment equations

$$f_\phi = \kappa\cos(\phi_t)\cos(\phi_{t-\tau})r_A(\tau) - \sigma^2\sin(\phi_t)\cos(\phi_t)\cos^2(\phi_{t-\tau})r_A(\tau)^2 - \frac{1}{4}\sigma^2\sin(2\phi_t)\cos(2\phi_t) - 1 + \\ + \sigma^2\sin(2\phi_t)\cos(\phi_t)\cos(\phi_{t-\tau})r_A(\tau) - \zeta\sin(2\phi_t) - \frac{1}{2}\kappa\cos(2\phi_t) - \frac{\kappa}{2} - \frac{1}{4}\sigma^2\sin(2\phi_t), \quad (26)$$

$$g_\phi = \sigma\cos(\phi_t)\cos(\phi_{t-\tau})r_A(\tau) - \sigma\cos^2(\phi_t), \quad (27)$$

$$f_A = \kappa\sin(\phi_t)\cos(\phi_{t-\tau})r_A(\tau) - \sigma^2\cos^3(\phi_t)\cos(\phi_{t-\tau})r_A(\tau) + \frac{1}{2}\sigma^2\cos^2(\phi_t)\cos^2(\phi_{t-\tau})r_A(\tau)^2 + \\ + \zeta\cos(2\phi_t) - \zeta - \frac{1}{2}\kappa\sin(2\phi_t) + \frac{1}{4}\sigma^2\cos(2\phi_t) + \frac{1}{16}\sigma^2\cos(4\phi_t) + \frac{3\sigma^2}{16}, \quad (28)$$

$$g_A = \sigma\sin(\phi_t)\left(\cos(\phi_{t-\tau})r_A(\tau) - \cos(\phi_t)\right). \quad (29)$$

Appendix B - Backwards Euler method

Notations for the Backwards Euler method for the stationary Fokker-Planck equation:

$$p_0 = p\left(\frac{\pi}{2}\right), \qquad p_n = p\left(\frac{\pi}{2} + n\Delta\phi\right), \quad (30)$$

where $\Delta\phi$ is a chosen resolution of the interval $[-\pi, \pi]$. The Backwards Euler method:

$$\frac{\partial}{\partial\phi}p_n \approx \frac{p_n - p_{n-1}}{\Delta\phi}, \quad (31)$$

For this special case, the p_n will be:

$$p_n = \frac{p_{n,\mathrm{numerator}}}{p_{n,\mathrm{denominator}}}, \quad (32)$$

where

$$p_{n,\mathrm{numerator}} = 8(p_{n-1}\sigma^2\cos(\phi_{t-\tau})r_A(\tau)\sin^2(n\Delta\phi)(\cos(\phi_{t-\tau})r_A(\tau) - \\ - 2\sin(n\Delta\phi)) + p_{n-1}\sigma^2\sin^4(n\Delta\phi) + 2p_0\Delta\phi), \quad (33)$$

$$p_{n,\mathrm{denominator}} = 8\cos(\phi_{t-\tau})r_A(\tau)\sin(n\Delta\phi)(\sigma^2\cos(\phi_{t-\tau})r_A(\tau)\sin(n\Delta\phi) - \\ - 2\kappa\Delta\phi + \sigma^2(\cos(2n\Delta\phi) - \Delta\phi\sin(2n\Delta\phi)) - \sigma^2) + 8\Delta\phi(\kappa+2) - 16\Delta\phi\zeta\sin(2n\Delta\phi) - \\ - 4(2\kappa\Delta\phi + \sigma^2)\cos(2n\Delta\phi) + \sigma^2(\cos(4n\Delta\phi) + 16\Delta\phi\sin^3(n\Delta\phi)\cos(n\Delta\phi)) + 3\sigma^2. \quad (34)$$