Energy Transport and Localization in the System of Harmonically Coupled Pendulums

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Summary. We provide an analysis of the nonlinear dynamics of the systems of two identical pendulums coupled via harmonic interaction. The presented analysis is valid for both low- and high-amplitude cases, even when the quasi-linear approximation cannot be applied. The described models are fundamental for many areas of Physics and Mechanics (paraffin crystals, DNA molecules etc.). We obtained the basic stationary solutions corresponding to nonlinear normal modes (NNMs). We have revealed an inversion of the NNM frequencies when the amplitude of the oscillations increases. Supposing the NNMs resonant interaction we introduce a "slow" time-scale which determines a characteristic time of the energy exchange between the pendulums. Introducing the angle variables we reduce the considered phase space. Essentially nonstationary process of the energy exchange is described in terms of the Limiting Phase Trajectories (LPTs), and we present its efficient analytical representation. We obtained also analytical criteria of the NNMs instability as well as the criteria of the transition from energy exchange to energy localization on the initially excited pendulum. The consideration of the asymptotic system in the "slow" time-scale is provided using the phase plane analysis and Poincare sections method. All analytical dependences are in a good correspondence with those obtained by numerical simulations. It is shown that these results can be extended to the finite chains of pendulums, and corresponding dependences are presented in analytical and graphical forms. We also compare our results with the behavior of the system of linearly coupled pendulums and denote a qualitative difference between these two models.

Dynamics of coupled nonlinear oscillators attracts the growing interest of scientific community because its fundamental meaning and various applications. In current paper, in contrast to many works devoted to interacting nonlinear oscillators, non-linearity of both the pendulums and the coupling between them are not assumed to be small. Thus, research methods that involve quasi-linearity and the presence of a small parameter characterizing nonlinearity and/or coupling are not applicable. To overcome this difficulty a semi-inverse method was proposed [1]. Using this method and LPT concept the system of two identical linearly coupled pendulums was examined under different oscillation amplitudes [2]. Stationary and non-stationary topological transitions leading to a qualitative change in the dynamic behaviour of the system were analytically described. This work continues the previous investigations for the more complex case, when the coupling between the pendulums cannot be assumed linear.

First, we introduce system of two identical coupled pendulums. In the dimensionless form the Hamiltonian reads:

$$H=\sum_{j=1,2}\left(\frac{1}{2}\left(\frac{dq_j}{dt}\right)^2+\left(1-\cos(q_j)\right)+\frac{1}{2}\beta\left(1-\cos(q_j-q_{3-j})\right)\right).$$

We introduce complex variables $\psi_j=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\omega}}\frac{dq_j}{dt}+i\sqrt{\omega}q_j\right)$ and suppose that two NNMs of the system are in a strong interaction. Using a semi-inverse method for asymptotic procedure, we introduce an effective small parameter, characterizing frequency difference between the two NNMs. In the main asymptotic approximation $\psi_j=\varphi_j e^{i\omega t}$, $j=1,2$, we obtain an analytic dependence for frequency on the amplitude of the initial excitation (Fig.1).

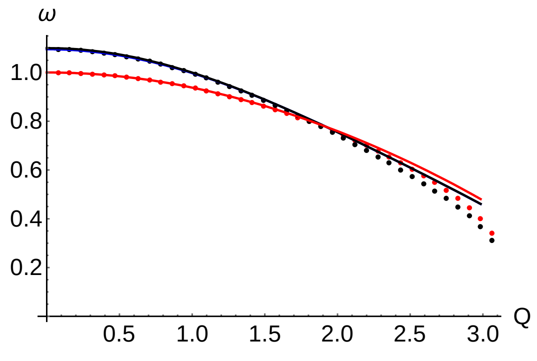

Figure 1 Comparison of the analytical solution with the numerical results for in-phase(black color) and antiphase(red color) modes. Solid lines define the analytical results, dotted lines correspond to results obtained numerically from the initial system, β=0.1

Taking in account additional integral of motion that appears in the slow-flow system and introducing angle variables $\varphi_1=\sqrt{X}\cos(\theta)\,e^{i\delta_1}$, $\varphi_2=\sqrt{X}\sin(\theta)\,e^{i\delta_2}$, $\Delta=\delta_1-\delta_2$, we obtain a system of two equations, which can be studied on the phase plane (θ, Δ) (see Fig.2). For the system in we have found analytical expressions for two different threshold values of the controlling parameter (coupling). The first value corresponds to the one of the NNMs stability loss. Another one corresponds to the coupling parameter value when the full energy transport transforms to energy localization if all the initial excitation is loaded on one of the pendulums. The analytical results are in a good agreement with those obtained from the Poincare sections of the initial system (see Fig.3). We have also found the analytical expressions for LPT of different types.

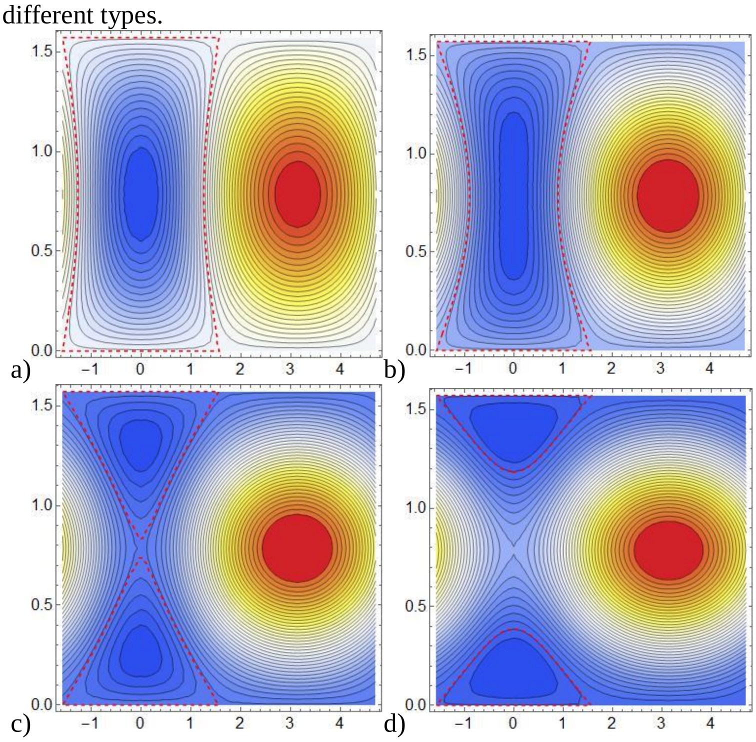

Figure.2 Phase plane (θ, Δ) for the system in the main asymptotic approximation, Q=π/2 and values of coupling parameter: a) β=0.3; b) β=0.139; c) β=0.0812; d) β=0.06. The LPT is denoted by red dashed line.

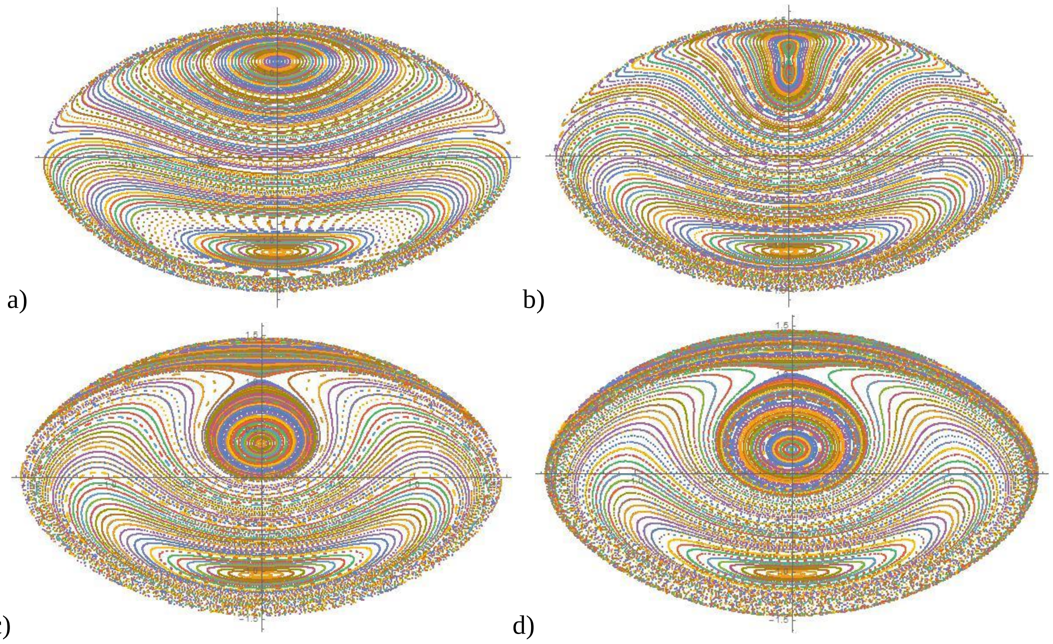

Figure 3. Poincare sections for the initial system, Q=π/2 and values of coupling parameter: a) β=0.3; b) β=0.139; c) β=0.0812; d) β=0.06

Conclusions

We report the study of two simmetrical systems of two harmonically coupled pendulums. We have studied NNMs, obtained analiticaly their frequency charachteristics and areas of stability. We have also formulated criteria for excistence of intensive energy transport between the pendulums and found analytical solutions for the LPT trajectories of different types. All the analytical results are in a good agreement with those by numerical simulations. Authors are thankful to the Russian Foundation for Basic Research for finantial support (Grant № 16-33-60186).

References

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