Analysis of Pivoting Algorithms for LCPs in Redundant Contact Dynamics

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Summary. This paper analyzes pivoting algorithms applied to solve linear complementarity problems (LCPs) in the context of redundant contact dynamics. The LCP formulation arising from unilateral constraints in multibody systems is presented. We describe the steps of two different pivoting algorithms: Lemke's and Murty's. Furthermore, we investigate the behaviour of these methods for redundant, frictionless n-legged frames on the ground. Lemke's algorithm always finds a solution whereas Murty's can run into a singularity. Each algorithm may obtain different solutions for the same contact problem.

Introduction

For modelling contact in multi-rigid-body systems, one can choose the constraint-based approach where interaction between two bodies is described by *unilateral constraints*. The *gap function* $\boldsymbol{\phi}$ measures the distance between the closest points of two rigid bodies. Nonnegativity $\boldsymbol{\phi} \geq \mathbf{0}$ ensures that there is no interpenetration. Due to these inequalities, the dynamics formulation cannot be expressed as a linear system but becomes a *linear complementarity problem* (LCP).

The LCP Formulation

The mathematical formulation of an LCP of size n is given by

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{b} &= \mathbf{w}, \\ \mathbf{0} \leq \mathbf{w} &\perp \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{1}$$

where the lead matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and the right-hand-side vector $\mathbf{b} \in \mathbb{R}^n$ are known. For admissible solutions, the vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ are constrained to be *feasible* (nonnegative) and *complementary* ("$\perp$"), i.e. x_i or w_i must be zero for all vector components i so that $w_i x_i = 0$. The dynamics formulation of a multibody system for the constraint-based approach using a finite difference approximation for the accelerations can be written as

$$\begin{gathered} \begin{bmatrix} \mathbf{M} & -\mathbf{J}_n^{\mathrm{T}} \\ \mathbf{J}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}^+ \\ h\boldsymbol{\lambda}_n^+ \end{bmatrix} + \begin{bmatrix} -\mathbf{M}\mathbf{v} - h\mathbf{f}_a \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \dot{\boldsymbol{\phi}}^+ \end{bmatrix}, \\ \mathbf{0} \leq \dot{\boldsymbol{\phi}}^+ \perp h{\boldsymbol{\lambda}_n}^+ \geq \mathbf{0}, \end{gathered} \tag{2}$$

which represents a mixed LCP at the velocity level [1] with mass matrix $\mathbf{M}$, constraint Jacobians $\mathbf{J}_n$, contact forces $\boldsymbol{\lambda}_n$, generalized velocities $\mathbf{v}$, time step size h, applied forces $\mathbf{f}_a$ and the time derivative of the gap function $\dot{\boldsymbol{\phi}}$. The superscript $^+$ denotes the values of the next time step. The mixed LCP can be transformed into LCP-form by substituting the unknown velocities $\mathbf{v}^+$ [2]. This is equivalent to forming $\mathbf{A} = \mathbf{J}_n\mathbf{M}^{-1}\mathbf{J}_n^{\mathrm{T}}$ and $\mathbf{b} = \mathbf{J}_n\mathbf{M}^{-1}(\mathbf{M}\mathbf{v} + h\mathbf{f}_a)$ in Eq. (1). Hence, $\mathbf{A}$ is symmetric. $\mathbf{x}$ and $\mathbf{w}$ are the contact impulses and the gap velocities, respectively. According to the complementarity condition, there are two possible configurations for contact i: If two bodies in contact move away from each other ($w_i \geq 0$), there must not be any additional impulse exchanged ($x_i = 0$). Conversely, an impulse must be applied ($x_i \geq 0$) in order to prevent two contacting bodies from moving toward each other ($w_i = 0$).

Pivoting Algorithms

Although solution algorithms for LCPs have been around for about half a century, the analysis of the physical meaning of the algorithm steps in the context of contact between rigid bodies does not exist in the literature to our knowledge. The objective of this paper is to analyze the algorithm behaviour in this context.

There are four types of algorithms for solving LCPs: *direct*, *iterative*, *enumerative* and *interior-point* methods, presented from a mathematical point of view in [3]. Our contribution focuses on direct methods, also known as *pivoting*, which systematically search for a solution to the LCP by interchanging variables between different sets. The complementarity condition permits two options of sets for each component i of $\mathbf{x}$: zero or nonnegative. This gives a total of 2^n different options. In this paper, we perform a step-by-step analysis for simple test cases for two of the most well-known pivoting algorithms: Lemke's algorithm [4] and Murty's *principal pivoting method* [5].

The Algorithm Steps

Murty's method performs a single principal pivot in each iteration, i.e. a variable is always pivoted with its complementary one, which leads to complementary, infeasible solutions at each substep. The effect of the pivot is not verified before performing it ("*pivot then check*"). The algorithm terminates when feasibility is reached and is proven to find a unique solution pair $(\mathbf{w}, \mathbf{x})$ for positive definite lead matrices [5, 6].

In Lemke's algorithm, an artificial variable x_0 is added if the LCP is not satisfied by the initial guess

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{c}x_0 + \mathbf{b} &= \mathbf{w}, \\ \mathbf{0} \leq \mathbf{w} \perp \mathbf{x} &\geq \mathbf{0},\ x_0 \geq 0, \end{aligned} \tag{3}$$

where $\mathbf{c} > \mathbf{0}$. After finding the "most negative" variable, x_0 is pivoted to be nonnegative so that Eq. (3) holds. In the following steps, Lemke's algorithm always produces feasible, almost-complementary solutions to the augmented problem (Eq. (3)). In each iteration, it calculates the maximum impulse/velocity under the condition that all variables remain nonnegative ("*check then pivot*") which corresponds to the minimum quotient rule in [3]. The method terminates when x_0 is pivoted to zero again and Eq. (1) is satisfied. Nonunique solutions for $\mathbf{x}$ are proven to exist for a positive semi-definite lead matrix $\mathbf{A}$. If $\mathbf{A}$ is also symmetric, like in our case, the solution for $\mathbf{w}$ is unique.

N-legged Frames on the Ground

We investigate the behaviour of the algorithms for processing frictionless contact problems involving a single dynamic body such as an n-legged frame on the static ground (see Fig. 1). Connecting the contact points by edges leads to a regular or irregular n-gon in the contact plane shown in Fig. 2. In general, external forces and torques are applied to the n-legged frame. The system is overdetermined for $n > 3$ legs and thus has no unique solution for $\mathbf{x}$. A system of forces and torques, including gravity, acting on a single rigid body can always be reduced to a resultant force-torque pair along a *line of action* (LOA).

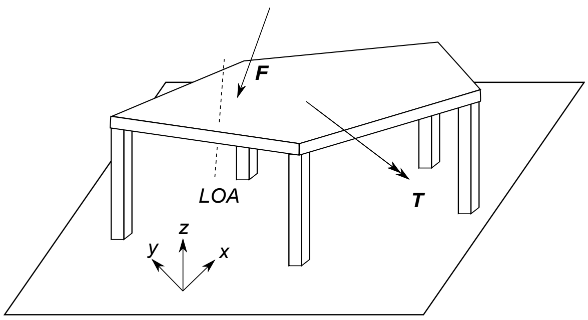

Figure 1: Irregular 5-legged-frame on the ground, line of action (LOA) of the resultant force-torque pair

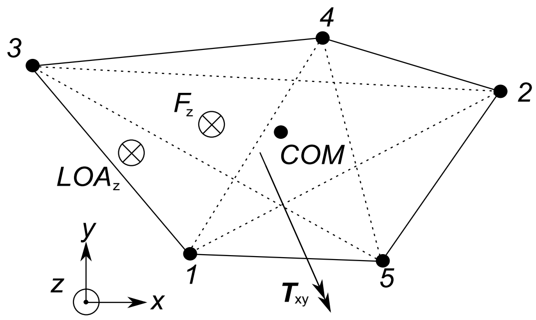

Figure 2: Topview of the corresponding 5-gon, line of action (LOA) shifted from the centre of mass (COM)

For redundant n-legged frames ($n > 3$), Lemke's algorithm distributes the contact impulses on at most three contact points. If the LOA does not intersect the plane, that is parallel to the contact plane and contains the centre of mass (COM), outside the polygon or on an edge connecting two contact points, exactly three contact points receive a nonzero impulse component x_i. In this case, there are at least $n-2$ different triangles illustrating the possibility of such a three-point solution for $\mathbf{x}$. For the pentagon in Fig. 2, these are 1-2-3, 1-5-3 and 1-4-3, so all triangles that include the intersection point with the line of action. A standard implementation of Lemke's algorithm [3] will obtain one of these $\mathbf{x}$-solutions. The vertex order and the rule for choosing the pivoting variable amongst several eligible ones determine the computed solution. Hence, applying different algorithm implementations to the same problem with different vertex orders can return different solutions for the contact impulses $\mathbf{x}$. However, the gap velocities $\mathbf{w}$ are always the same due to w-uniqueness for symmetric positive semi-definite lead matrices [6]. Physically, this is necessary to obtain the same motion in all cases. Depending on the initial guess and the problem, Murty's method may not find a solution which is not surprising due to the semi-definiteness of the lead matrix. If it does, it also picks at most three contact points so that the nonuniqueness of the solution applies as well, just like for Lemke's algorithm.

Conclusion

Lemke's algorithm chooses the pivoting candidate by comparing the effect of a pivot before performing it. Furthermore, it does not run into a singularity due to inversion of a rank-deficient matrix and always terminates, even for degenerate cases such as the n-legged frames. In case of degeneracy, the contact impulses found are not unique. One possibility to avoid different solutions for different algorithm implementations or problem formulations could be to choose the most evenly distributed contact impulses. Murty's algorithm pivots an infeasible variable without previously verifying the effect of the pivot. It is able to solve some cases, but there is no guarantee that the matrix to be inverted is always full rank.

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