On some extension of center manifold method

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Summary. We propose an asymptotic integration method for certain class of functional differential systems. This class includes the delay differential equations with oscillatory decreasing coefficients and variable delays that are close to constants at infinity. Both the ideas of the centre manifold theory and the averaging method together with some classical asymptotic theorems are used to construct the asymptotics for solutions. We illustrate the asymptotic integration method by constructing the asymptotics for solutions of two scalar delay differential equations.

Problem statement

We study the asymptotic integration problem for the functional differential system (FDS)

$$\dot{x} = B_0 x_t + G(t, x_t) \tag{1}$$

as $t \to \infty$. Here $x \in \mathbb{C}^m$, $x_t(\theta) = x(t+\theta)$ $(-\tau \le \theta \le 0)$ denotes the element of C_τ, where $C_\tau \equiv C\big([-\tau, 0], \mathbb{C}^m\big)$ is the set of all continuous functions defined on $[-\tau, 0]$ and acting to $\mathbb{C}^m$. We consider Eq. (1) as perturbation of linear autonomous system

$$\dot{x} = B_0 x_t, \tag{2}$$

where B_0 is a bounded linear functional acting from C_τ to $\mathbb{C}^m$ that does not depend on t. Linear bounded functional $G(t, \cdot)$, acting from C_τ to $\mathbb{C}^m$, is, in some sense, a small perturbation. The detailed structure of the functional $G(t, \cdot)$ will be defined later. The main assumption concerning the unperturbed Eq. (2) is the following. The characteristic equation

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda I - B_0(e^{\lambda\theta} I),$$

has N roots (with account of their multiplicities) $\lambda_1, \dots, \lambda_N$ with zero real parts and all other roots have negative real parts. This makes possible to use the ideas of the center manifold theory (see, e.g., [1–4]) for asymptotic integration of Eq. (1). The asymptotic integration problem for FDS having form (1) was studied by the author in papers [8, 9]. In [8], the method for asymptotic integration of Eq. (1), where $B_0 = 0$, was proposed. The more general results were obtained in [9], where the center manifold technique was adapted to construct the asymptotics for solutions of Eq. (1). In this talk we briefly describe the obtained results and show how we can extend the developed asymptotic integration method to a wider class of FDS. We demonstrate this method by constructing the asymptotic formulas as $t \to \infty$ for solutions of the delay differential equation

$$\dot{x} = -\frac{\pi}{2} x(t-1) + \frac{a \sin \omega t}{t^\rho} x(t-h) \tag{3}$$

and solutions of the differential equation with variable delay

$$\dot{x} = -\frac{\pi}{2} x\Big(t - 1 + \frac{a \sin \omega t}{t^\rho}\Big). \tag{4}$$

In Eqs. (3), (4) we assume that $a, \omega \in \mathbb{R}\backslash\{0\}$, $h \ge 0$ and $\rho > 0$.

Asymptotic integration method

We begin this section by clarifying the form of the functional $G(t, \cdot)$ in Eq. (1). Namely, we assume that

$$G(t, x_t) = B(t, x_t) + R(t, x_t), \tag{5}$$

where $B(t, \cdot)$ and $R(t, \cdot)$ are linear bounded functionals acting from C_τ to $\mathbb{C}^m$. Besides, for each $\varphi \in C_\tau$ function $R(\cdot, \varphi)$ is Lebesgue measurable for $t \ge t_0$ and $|R(t, \varphi)| \le \gamma(t) \|\varphi\|_{C_\tau}$, where $\gamma(t) \in L_1[t_0, \infty)$. Moreover, for each infinitely differentiable function $\varphi(\theta)$ the following representation holds:

$$B(t, \varphi) = \sum_{i=1}^{n} v_i(t) P_i^\varphi(t) + \dots + \sum_{1 \le i_1 \le \dots \le i_k \le n} v_{i_1}(t) \cdots v_{i_k}(t) P^\varphi_{i_1 \dots i_k}(t) + R^\varphi(t). \tag{6}$$

Here $P^\varphi_{i_1 \dots i_l}(t)$ are certain vector-valued trigonometric polynomials, depending on function $\varphi(\theta)$. Further, $v_1(t), \dots, v_n(t)$ are scalar absolute continuous on $[t_0, \infty)$ functions, satisfying the following conditions:
1^0. $v_1(t) \to 0, v_2(t) \to 0, \dots, v_n(t) \to 0$ as $t \to \infty$;
2^0. $\dot{v}_1(t), \dot{v}_2(t), \dots, \dot{v}_n(t) \in L_1[t_0, \infty)$;
3^0. There exists $k \in \mathbb{N}$ such that $v_{i_1}(t) v_{i_2}(t) \cdots v_{i_{k+1}}(t) \in L_1[t_0, \infty)$ for any sequence $1 \le i_1 \le i_2 \le \dots \le i_{k+1} \le n$.
Finally, the vector function $R^\varphi(t)$ in (6) belongs to $L_1[t_0, \infty)$. Thus, for each infinitely differentiable function $\varphi(\theta)$ the

vector function $B(\cdot, \varphi)$ consists of two terms: the first one has an oscillatory decreasing form as $t \to \infty$ and the second one is a certain depending on $\varphi(\theta)$ absolutely integrable on $[t_0, \infty)$ function.
The essence of the proposed asymptotic integration method is to construct the center-like manifold (also called critical manifold) for Eq. (1), (5) which is positively invariant for trajectories of Eq. (1) for $t \geq t_*$ and attracts all the trajectories of Eq. (1) for sufficiently large t. Critical manifold for Eq. (1) can be analytically defined as the set

$$\mathcal{W}(t) = \Big\{\varphi(\theta) \in C_\tau \;\Big|\; \varphi(\theta) = \Phi(\theta)u + H(t,\theta)u,\ u \in \mathbb{C}^N\Big\}.$$

Here $\Phi(\theta)$ is $(m \times N)$-matrix whose columns are the generalized eigensolutions of the unperturbed Eq. (2) corresponding to the eigenvalues $\lambda_1, \dots, \lambda_N$ with zero real parts. Further, $H(t,\theta)$ is a certain continuous in $t \geq t_*$ and $\theta \in [-\tau, 0]$ $(m \times N)$-matrix such that $\|H(t,\cdot)\|_{C_\tau} \to 0$ as $t \to \infty$, where $\|H(t,\cdot)\|_{C_\tau} = \sup\limits_{-\tau \leq \theta \leq 0} |H(t,\theta)|$ and $|\cdot|$ is some matrix norm. As a rule we can not define matrix $H(t,\theta)$ explicitly. Nevertheless, it can be approximated by a certain explicitly defined matrix $\hat{H}(t,\theta)$ up to the term $Z(t,\theta)$ such that $\|Z(t,\cdot)\|_{C_\tau} \to 0$ as $t \to \infty$ and $\|Z(t,\cdot)\|_{C_\tau} \in L_1[t_*, \infty)$. The main properties of critical manifold $\mathcal{W}(t)$ and the method for its approximate construction are discussed in paper [9].
Evidently, $\mathcal{W}(t)$ is N-dimensional linear space. It turns out that, due to the positive invariance of $\mathcal{W}(t)$, the dynamics of solutions of Eq. (1) lying on this manifold can be described by the N-dimensional linear ordinary differential system

$$\dot{u} = \Big[D + \Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big)\Big]u, \qquad t \geq T. \tag{7}$$

Here D is a certain $(N \times N)$-matrix whose eigenvalues are $\lambda_1, \dots, \lambda_N$ and $\Psi(0)$ is a certain $(N \times m)$-matrix coming from decomposition of C_τ (see, e.g., [6]). System (7) is referred to as a projection of Eq. (1) on critical manifold $\mathcal{W}(t)$ or, simply, as a system on critical manifold. Since $\mathcal{W}(t)$ is an attractive manifold it can be shown that for each solution $x(t)$ of Eq. (1) the following asymptotic formula holds as $t \to \infty$: $x_t(\theta) = \Phi(\theta)u_H(t) + H(t,\theta)u_H(t) + O\big(e^{-\beta t}\big)$. Here $u_H(t)$ is a certain solution of Eq. (7) and $\beta > 0$ is a certain real number. Therefore, to obtain the asymptotics for all solutions of Eq. (1) we should construct the asymptotic formulas for the fundamental solutions $u^{(1)}(t), \dots, u^{(N)}(t)$ of a system on critical manifold (7). This can be done by using the averaging technique from [7] together with the well-known Levinson's theorem (see, e.g., [5]). Finally, we obtain the following asymptotic representation for solutions of Eq. (1):

$$x(t) = x_t(0) = \big(\Phi(0) + H(t,0)\big)\sum_{i=1}^{N} c_i u^{(i)}(t) + O\big(e^{-\beta t}\big), \qquad t \to \infty,$$

where $c_1, \dots, c_N$ are arbitrary complex constants and $\beta > 0$ is a certain real number.

Conclusion

We use the described above method to construct the asymptotics for solutions of Eq. (3) and Eq. (4) as $t \to \infty$. If $\rho > 1$, the dynamics of solutions of Eq. (3) is the same as the dynamics of solutions of Eq. (4). The asymptotics of solutions is described in this case by the formula

$$x(t) = c_1\big(1 + o(1)\big)e^{\mathrm{i}\frac{\pi}{2}t} + c_2\big(1 + o(1)\big)e^{-\mathrm{i}\frac{\pi}{2}t} + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary complex constants and $\beta > 0$ is a certain real number. If $1/2 < \rho \leq 1$, the dynamics of solutions of Eqs. (3), (4), being qualitatively the same, have some quantitative differences. If $\rho \leq 1/2$, the dynamics of solutions of Eq. (3) and the dynamics of solutions of (4) as $t \to \infty$ significantly differ. These are the results that will be discussed in the talk.

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References

[1] Ait Babram M., Hbid M.L., Arino O. (1997) Approximation Scheme of a Center Manifold for Functional Differential Equations. *J. Math. Anal. Appl.* **213**:554-572.

[2] Arino O., Hbid M.L., Ait Dads E. (Eds.) (2006) Delay Differential Equations and Applications. Springer, Dordrecht.

[3] Balachandran B., Kalmár-Nagy T., Gilsinn D.E. (Eds.) (2009) Delay Differential Equations: Recent Advances and New Directions. Springer, New York.

[4] Carr J. (1981) Applications of Centre Manifold Theory. Springer-Verlag, New York.

[5] Eastham M.S.P. (1989) The Asymptotic Solution of Linear Differential Systems. Clarendon Press, Oxford.

[6] Hale J., Verduyn Lunel S.M. (1993) Introduction to Functional Differential Equations. Springer-Verlag, New York.

[7] Nesterov P.N. (2007) Averaging Method in the Asymptotic Integration Problem for Systems with Oscillatory-Decreasing Coefficients. *Differ. Equ.* **43**:745–756.

[8] Nesterov P. (2013) Asymptotic Integration of Functional Differential Systems with Oscillatory Decreasing Coefficients. *Monatsh. Math.* **171**:217-240.

[9] Nesterov P. (2016) Asymptotic Integration of Functional Differential Systems with Oscillatory Decreasing Coefficients: a Center Manifold Approach. *Electron. J. Qual. Theory Differ. Equ.* **33**:1-43.