Uncovering detached resonance curves in single degree-of-freedom systems

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Summary. The problem of the existence of detached resonance curves (DRCs) in smooth single degree-of-freedom systems is addressed. Using singularity theory in conjunction with a classical harmonic balance procedure, conditions of existence of DRCs are analytically defined for systems encompassing either nonlinear damping or elastic force. The obtained results shed light on the mechanisms leading to the appearance of DRCs. Interestingly, the properties required for the generation of DRCs resemble those exhibited by common physical models, such as the Stribeck friction force and the softening-hardening behavior of buckled structures, namely they are characterized by non-monotonically increasing functions.

Introduction

The identification of detached resonance curves (DRCs) is particularly troublesome. Well-established continuation techniques are generally incapable of tracking them and also global numerical investigation might fail to catch them if not carried out thoroughly. Experimental investigations performed through frequency sweeps are similarly unable to identify DRCs. Recent studies showed that DRCs are a relatively common feature of nonlinear systems and are usually related to internal resonances [1–5], but they can also be ascribed to subharmonic resonances in nonsmooth systems [6, 7]. They can lie outside or inside the main resonance curve [1]. In [8], a technique, based on nonlinear normal modes, for the identification of DRCs related to internal resonances was developed.

In this study, conditions for the existence of DRCs in smooth single degree-of-freedom (DoF) systems are investigated. The analytical investigation exploits the fact that generation and disappearance of DRCs correspond to singular points in the frequency response function. Thus, conditions for the presence of these singularities [9] when coupled with a harmonic balance technique yield conditions for the existence of DRCs. Two different systems are considered, one encompassing nonlinear elastic force and the other one possessing nonlinear damping force.

Mathematical models and analytical procedure

Adopting dimensionless parameters, the dynamics of the two systems is described by

$$\ddot{x} + x + c_1\dot{x} + k_3x^3 + x^5 = 2f\cos(\omega t) \quad \text{and} \quad \ddot{x} + x + c_1\dot{x} + c_3\dot{x}^3 + \dot{x}^5 = 2f\cos(\omega t)\,, \tag{1}$$

where c_1 is the linear damping coefficient, c_3 and k_3 are the cubic damping and resotoring force coefficients, respectively, and f is the forcing amplitude. In both cases, the nonlinear functions are characterized by a linear, a cubic and a quintic term. Linear and quintic terms are assumed positive, such that the system trivial position remains stable and the system does not diverge to infinite for too large oscillation amplitudes. In order to transform the nonlinear differential equations into nonlinear algebraic equations, we apply the harmonic balance procedure, i.e. we approximate the particular solution to a single harmonic, $x = ae^{j(\omega t+\phi)} + ae^{-j(\omega t+\phi)}$. After some algebraic manipulations we obtain for the two systems

$$\begin{aligned} g_1 &= 100A^5 + 60A^4k_3 + A^3\left(9k_3^2 - 20\Omega + 20\right) - 6A^2k_3(\Omega - 1) + A\left(\left(c_1^2 - 2\right)\Omega + \Omega^2 + 1\right) - F = 0 \\ g_2 &= 100A^5\Omega^5 + 60A^4c_3\Omega^4 + A^3\Omega^3\left(20c_1 + 9c_3^2\right) + 6A^2c_1c_3\Omega^2 + A\left(\left(c_1^2 - 2\right)\Omega + \Omega^2 + 1\right) - F = 0, \end{aligned} \tag{2}$$

where $A = a^2$, $\Omega = \omega^2$ and $F = f^2$. Solutions of $g_1 = 0$ and $g_2 = 0$ approximate the frequency response of the two systems under study.

The diagram described by Eq. (1a) (or Eq. (1b)) presents isola singularities or simple bifurcations when the following conditions are verified [9]

- isola

$$g_1 = \frac{\partial g_1}{\partial \Omega} = \frac{\partial g_1}{\partial A} = 0, \quad \frac{\partial^2 g_1}{\partial A^2} \neq 0, \quad \det\left(\mathrm{d}^2 g_1\right) > 0, \tag{3}$$

- simple bifurcation

$$g_1 = \frac{\partial g_1}{\partial \Omega} = \frac{\partial g_1}{\partial A} = 0, \quad \frac{\partial^2 g_1}{\partial A^2} \neq 0, \quad \det\left(\mathrm{d}^2 g_1\right) < 0, \tag{4}$$

where $\mathrm{d}^2 g_1$ is the Hessian matrix of $g_1(A, \Omega)$. Isola singularities characterize the appearance of DRCs while simple bifurcations are associated to the merging of two different branches.

Exploiting conditions in Eqs. (3) and (4) and the approximate descriptions of the frequency responses g_1 and g_2, we determine for which parameter values DRCs exist. According to our analysis, for both systems, DRCs are generated when the third order nonlinear coefficient, k_3 or c_3, is negative and its absolute value sufficiently large, which causes the nonlinear function to be non-monotonically increasing.

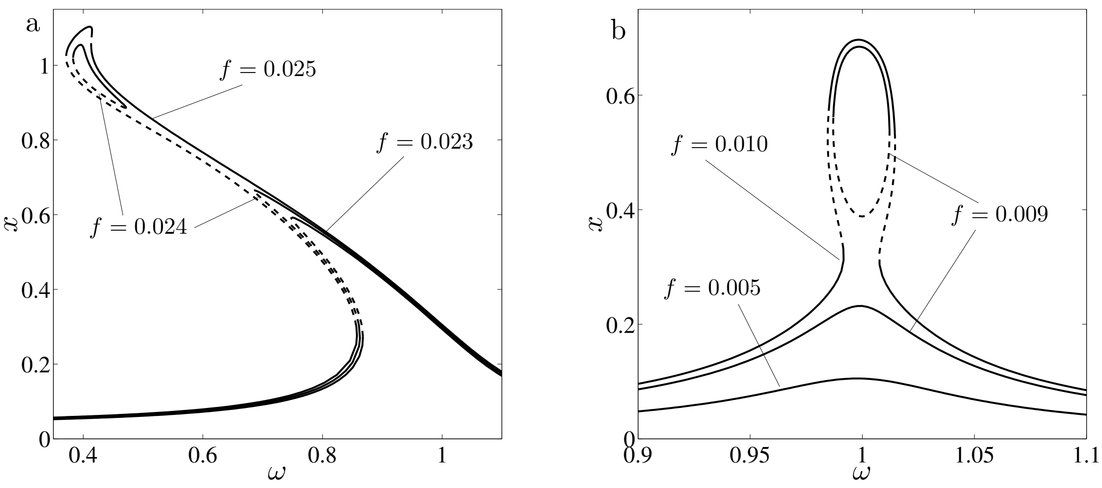

Figure 1: Frequency response for the systems in Eqs. (1a) and (1b) (In (a) and (b), respectively). Parameter values: (a) $c_1 = 0.1$, $k_{nl3} = -1.9$ and (b) $c_1 = 0.1$, $c_3 = -0.6$; force as indicated in the plots.

Numerical validation and physical interpretation

Figure 1a depicts the appearance of a DRC for a system encompassing a nonlinear elastic force. For $f = 0.023$ the system presents no DRC, increasing the forcing amplitude, for $f = 0.024$, a DRC appears consequently to an isola singularity, then, for $f = 0.025$, it merges with the main resonant branch in correspondence of a simple bifurcation singularity. The system exhibits a strong softening due to the negative cubic term ($k_3 = -1.9$), while the resonant curve tends to bend back to the right for oscillation amplitudes larger than one. Although the adopted procedure does not suggest any physical interpretation of the phenomenon, we conjecture that the particular shape of the potential energy, which in correspondence of the appearance of the DRC has a very low slope, is somehow related to the generation of DRCs.

Figure 1b illustrates the generation of an isola for the system encompassing nonlinear damping force. The DRC is generated for $f \approx 0.006$ in correspondence of an isola singularity, while it merges with the main branch for $f \approx 0.0095$ through a simple bifurcation singularity. The phenomenon is probably related to the non-monotonically trend of the damping force with respect to velocity. In fact, higher amplitude motions may present a lower level of dissipation than motions with lower amplitude, with the consequent generation of DRCs.

The features characterizing a damping force that causes DRCs can be found in several models of dry friction forces, as for example the Stribeck friction model. Regarding nonlinear elastic force, a typical phenomenon that causes non-monotonic behavior is the buckling of slender structures. The elastic force of such systems resembles the one studied here, although buckling has its non-monotonic characteristic only in compression.

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