

The emergence and breaking of synchrony in networks of dynamical systems

Eddie Nijholt*, Bob Rink* and Jan Sanders*

*Department of Mathematics, VU University Amsterdam, 1081 HV Amsterdam, The Netherlands

Summary. Networks of coupled nonlinear dynamical systems often display unexpected phenomena. They may for example synchronise. This form of collective behaviour occurs when the agents of the network behave in unison. An example is the simultaneous firing of neurons. It has also been observed that network synchrony often emerges or breaks through quite unusual bifurcation scenarios. I will show how these phenomena can be understood from what we call “hidden symmetry”. The long term aim is a classification of synchrony-breaking bifurcations in networks in terms of indecomposable representations of hidden symmetries.

Coupled cell networks

Many systems throughout the sciences and in technology are networks: they are formed of individual nodes which interact. Examples include electrical circuits, Josephson junction arrays, power grids, neuronal networks, metabolic pathways, genetic regulatory networks, food chains, economic markets and social media. Such dynamical systems typically consist of nonlinear maps or differential equations which determine the behaviour of the individual nodes and the interactions between them. However, knowledge of the dynamics of a single isolated node and the interaction topology (determined, for example, by experiment), is typically not sufficient for understanding the global dynamics of the entire network. In fact, dynamical networks can exhibit intriguing emergent phenomena such as synchronization (in phase or frequency), either of the entire network or just of a localized subset of nodes. It is seldom clear how the interaction structure of a network determines its often peculiar dynamical behaviour. As a result, it is often unknown what is the functional relevance of connection configurations that are frequently observed in nature. Examples of this are the small “motifs” that are found with high statistical significance in (large) transcription regulatory networks.

Synchronization is one of the most distinctive emergent phenomena in network dynamics. This form of collective behaviour, that is not found in generic dynamical systems, is characterised by nodes behaving in unison. Examples include the simultaneous firing of neurons, bird flocks and social cooperation. An extensive theoretical framework for the study of exact phase synchrony was developed by Golubitsky, Stewart and coworkers around 2005. These authors define a class of networks (so-called “coupled cell networks”), to which one can associate a class of network dynamical systems (i.e. differential equations or iterated maps with a network structure). Their theory moreover provides a combinatorial characterisation of the synchrony patterns that these networks can support, and it has been successfully applied for example in the study of animal gaits and binocular rivalry. In 2015, the theory of Golubitsky and Stewart was cast in the language of category theory and graph fibrations by DeVille and Lerman.

Coupled cell networks can be defined mathematically as coloured directed graphs (consisting of cells and arrows) satisfying a number of natural conditions. For every coupled cell network one may define a class of dynamical systems. This is done by assigning to each cell in the network a dynamical variable. The arrows in the coupled cell network then determine the interactions between these variables, and the colours distinguish different types of variables and interactions. The precise definition of coupled cell networks and their associated network dynamical systems is straightforward but too long to provide here. Instead, let us illustrate the theory through the two simple networks **A** and **B** in Figure 1. Networks **A** and **B** consist of three identical cells and we assume that the states of these cells are determined by variables $x_1, x_2, x_3 \in \mathbb{R}$. Network dynamical systems are then defined by the ordinary differential equations that are also shown in Figure 1. These differential equations depend on the network graph and on a model-specific or even completely unknown response function $f = f(X_1, X_2, X_3)$. Accordingly, each cell evolves under the impact of its own state (first argument of f), the state of its solid blue input (second argument of f) and the state of its dashed red input (third argument of f). One could think of **A** and **B** as small networks of (groups of) identical neurons, each receiving for instance one excitatory signal (say the solid blue arrow) and one inhibitory signal (the dashed red arrow). The x_i would then be the membrane potentials of the neurons.

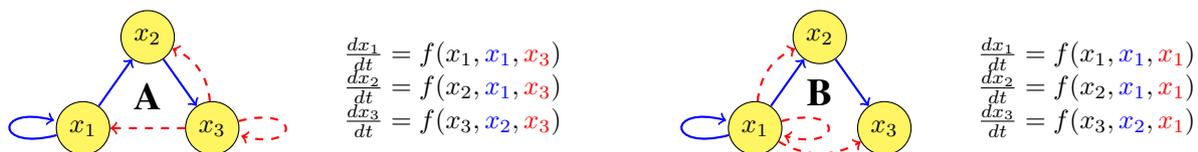


Figure 1: Two different networks with three identical cells and their equations of motion.

Synchrony and synchrony-breaking

Interestingly, setting $x_1 = x_2 = x_3$ in the differential equations for the networks yields that $\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dx_3}{dt}$. Both networks therefore admit an invariant subspace of *fully synchronous* solutions for which $x_1(t) = x_2(t) = x_3(t)$. For a similar reason there is also a subspace of *partially synchronous* solutions with $x_1(t) = x_2(t) \neq x_3(t)$. These so-called “synchrony patterns” do not depend on the precise form of f , but are forced by the network structure. Synchrony patterns

are thus dynamical invariants of networks. It is well understood why they exist, and it is relatively straightforward to compute them.

On the other hand, there are few results that explain how the architecture of a network dictates its non-synchronous dynamics. In particular, there are currently no tailored methods to predict how synchronous solutions in networks are born or break, or to demonstrate how network structure gives rise to complicated (e.g. chaotic) dynamical behaviour. For instance, when a parameter of a network is varied, synchronous and non-synchronous solutions may emerge in the dynamics through very different bifurcation scenarios in different networks. It is unclear which properties of a network determine these scenarios, and whether the bifurcations found in one network should also be expected in others. To illustrate this phenomenon, let us assume that the response function f of networks **A** and **B** depends on a parameter $\lambda \in \mathbb{R}$, and that the networks admit a fully synchronous steady state (i.e. a solution to $\frac{dx}{dt} = 0$). Then less synchronous steady states may emerge in a *synchrony-breaking bifurcation* as λ varies. It turns out that in a generic synchrony-breaking bifurcation, two branches of steady states are born from the synchronous state. Table 1 shows the main characteristics (synchrony type, growth rate and dynamical stability) of these branches (the so-called “branching pattern”). The table reveals that network **A** has the function of a “switch” and network **B** that of an “amplifier”. This difference between the networks is a delicate nonlinear effect that is not caused by the presence of synchrony alone (because the synchrony patterns of both networks are the same).

Network A		Network B	
Growth rate	Synchrony	Growth rate	Synchrony
$x_1 = x_2 = x_3 = 0$	Full	$x_1 = x_2 = x_3 = 0$	Full
$x_1 = x_2 \sim \lambda, x_3 \sim \lambda, x_{1,2} - x_3 \sim \lambda$	Partial	$x_1 = x_2 = 0, x_3 \sim \lambda$	Partial
$x_1 \sim \lambda, x_2 \sim \lambda, x_3 \sim \lambda$	None but almost partial	$x_1 = 0, x_2 \sim \lambda, x_3 \sim \pm\sqrt{\lambda}$	None
$x_1 - x_2 \sim \lambda^2, x_{1,2} - x_3 \sim \lambda$			

Table 1: Growth rate and synchrony type of steady state branches that emerge in a generic synchrony-breaking bifurcation near $(x, \lambda) = (0, 0)$ in networks **A** and **B**. Only the branches highlighted in green can be dynamically stable.

Hidden symmetry

It turns out that the synchrony-breaking bifurcations of the networks **A** and **B** are determined by *hidden symmetry*. Note that the networks clearly do not admit any classical symmetries (by permutations of cells). Nevertheless, there exists a nontrivial graph morphism from network **B** to itself that sends cells 1 and 2 to cell 1, and cell 3 to cell 2. We may think of this morphism as a (nonclassical because noninvertible) “symmetry” of network **B**, and it turns out that this symmetry is what forces the remarkable synchrony-breaking bifurcation of network **B**. Network **A** does not admit such a nontrivial morphism, but we discovered that network **A** can be embedded in a network with nonclassical symmetries (namely network $\tilde{\mathbf{A}}$ depicted in Figure 2). We think of the symmetries of $\tilde{\mathbf{A}}$ as *hidden symmetries* of **A**. One can also prove that the synchrony-breaking bifurcations reported in Table 1 are the only generic one-parameter bifurcations for differential equations with the (hidden) symmetries of networks **A** and **B**. It is thus the hidden symmetry of networks **A** and **B** that forces the bifurcations to have the properties of Table 1.

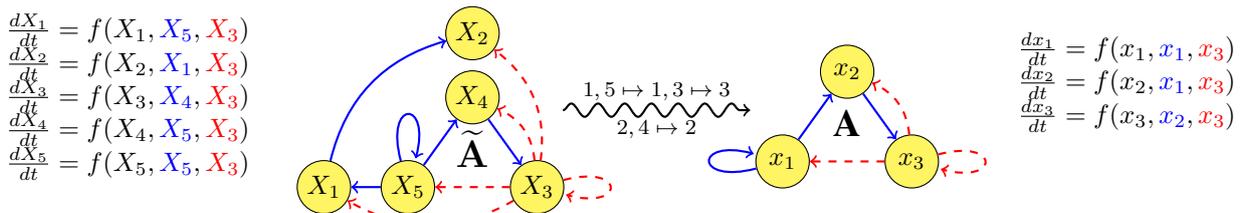


Figure 2: The dynamics of network **A** is embedded in that of network $\tilde{\mathbf{A}}$ as the synchrony pattern $X_1 = X_5 = x_1, X_2 = X_4 = x_2, X_3 = x_3$. Network $\tilde{\mathbf{A}}$ admits five nonclassical symmetries. One of these sends cell 1 to cell 5, cell 2 to cell 1, and fixes the remaining cells.

In my talk, it will be explained that every coupled cell network can be embedded in a network with nonclassical symmetry (i.e. that every coupled cell network has hidden symmetries). I will then show how exactly hidden symmetry governs the nonlinear properties of synchrony-breaking bifurcations. In particular, I will show that hidden symmetry can be preserved in some of the most important local reduction methods of dynamical systems theory (such as center manifold reduction). This has already led to a classification of synchrony-breaking bifurcations a number of interesting classes of networks.

References

- [1] B. Rink and J.A. Sanders, *Amplified Hopf bifurcations in feed-forward networks*, SIAM J. Appl. Dyn. Syst. 12 (2013), no. 2, 1135-1157.
- [2] B. Rink and J.A. Sanders, *Coupled cell networks and their hidden symmetries*, SIAM J. Math. Anal. 46 (2014), no. 2, 1577-1609.
- [3] B. Rink and J.A. Sanders, *Coupled cell networks: semigroups, Lie algebras and normal forms*, Trans. Amer. Math. Soc. 367 (2015), 3509-3548.
- [4] E. Nijholt, B. Rink and J.A. Sanders, *Graph fibrations and symmetries of network dynamics*, J. Differ. Equations 261 (2016), no. 9, 4861-4896.
- [5] E. Nijholt, B. Rink and J.A. Sanders, *Projection blocks in homogeneous coupled cell networks*, to appear in Dynamical Systems (2017).
- [6] E. Nijholt, B. Rink and J.A. Sanders, *Center manifolds of coupled cell networks*, arXiv 1603.08888 (2016).