# Parametrically excited inertial sensors

S. Amir Mousavi Lajimi* and Eihab Abdel-Rahman**
* *Motion Research Group, Faculty of Engineering, University of Waterloo, Canada, N2L 3G1*
** *Department of Systems Design Engineering, University of Waterloo, Canada, N2L 3G1*

*Summary*. A parametrically excited microsystem including a base beam and a pair of sense electrodes to measure the induced displacement is studied in this work. The excitation frequency is near twice the effective natural frequency of the cantilever creating principal parametric resonance. By using analytical and numerical methods, the response is characterized and demonstrated in frequency-response and calibration curves. An application of the sensor as a single-axis inertial sensor to measure acceleration is investigated in this work. Ultimately the design of the inertial sensor is improved by studying the nonlinear dynamics of the structure.

## Mathematical methods

The extended Hamilton's principle is used to derive the equation of motion, Galerkin's method is employed to obtain a single-mode approximation of the equation of motion, and the method of multiple scales is used to solve the single-mode approximation model and obtain the response amplitude and phase equations. Two cases are studied where the desired quantity, *i.e.* the acceleration, appears along, see Figure1a, and normal to, see Figure1b, the excitation force.

### Equation of motion

The structure includes a base beam (the driving element) and a secondary beam (the sensing element). For both cases, see Figure1a and Figure1b, the sense beam's motion is detected by employing two side electrodes creating a differential sense capacitor. We present our results in terms of displacement to provide an immediate understanding of the motion with respect to the initial gap between the electrodes and the moving mass (the cantilever).

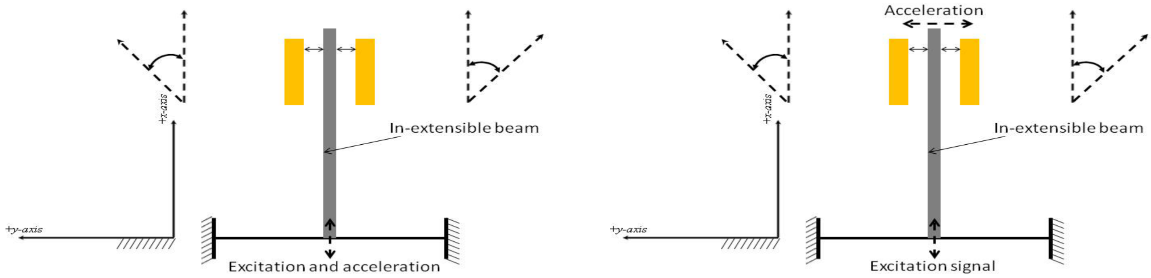



Figure 1: Cantilever beam under a parametric excitation (a) the excitation and the external acceleration appear along each other and (b) the excitation and the external acceleration act perpendicular to each other .

### Case 1

The motion of the base beam is characterized by its amplitude and frequency. The fringing field effects are taken into consideration given that the beam is considerably exposed to the electrode. We consider Crespo da Silva and Glynn's [1] fundamental mathematical model of a cantilever including the geometric nonlinearity. Following Lajimi and Heppler [2], the axial force denoting the linearly varying axial load for a uniform beam is replaced with

$$\hat{p}(\hat{s},\hat{t}) = \left(\hat{M} + \hat{m}(\hat{l} - \hat{s})\right)\left(\hat{a}_e - \hat{\omega}_b^2\,\hat{a}_b\,\cos(\hat{\omega}_b\,\hat{t})\right) \tag{1}$$

Furthermore, the electrostatic force acts in transverse direction on areas where the beam is between the two electrodes including the fringing field effect. We are interested to investigate the case where the sense electrodes are not loaded with an AC signal and therefore the output voltage is exclusively affected by the motion. Combining Crespo da Silva and Glynn's [1], Anderson et al. [3]'s, and Lajimi and Heppler's [2] mathematical models, for the beam under an axial force and a base excitation including a quadratic damping, we obtain the equation of motion as

$$\begin{aligned} v'''' + \ddot{v} =& -\alpha\,\dot{v} - \beta\,\dot{v}|\dot{v}| - d^2\left(4v'v''v''' + \left(v''^3 + v'^2 v''''\right)\right) - \frac{1}{2}d^2\left[v'\int_1^s\left[\frac{\partial^2}{\partial t^2}\int_0^s v'^2\,\mathrm{d}s\right]\mathrm{d}s\right]' \\ &+ \left(a_e - a_b\cos\left(\hat{\omega}_b\,\hat{t}\right)\right)\left(v' - (1-s)v''\right) + \hat{V}(\hat{t})^2\left(\frac{\nu + \nu_f\,(1-v)}{(1-v)^2} - \frac{\nu + \nu_f\,(1+v)}{(1+v)^2}\right)H(s,t) \end{aligned} \tag{2}$$

in nondimensional form. Using Galerkin's method a single-mode approximation of the equation of motion is obtained and solved using a perturbation method.

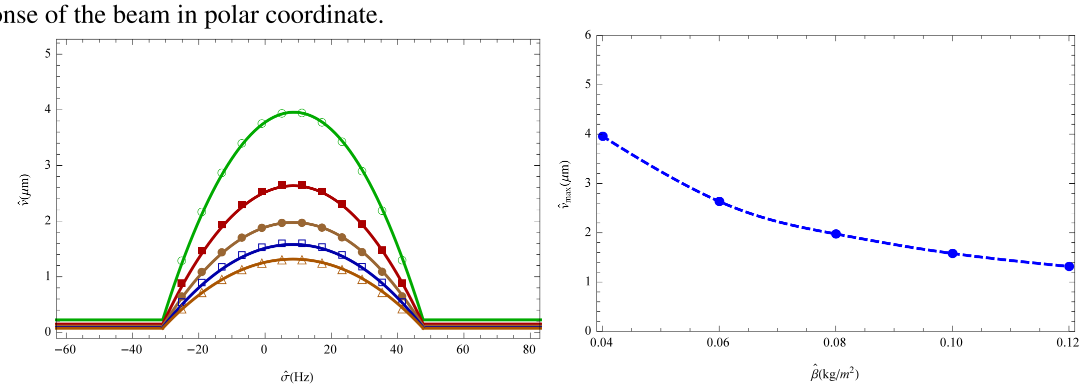
*amir.lajimi@uwaterloo.ca

**Case 2**

The difference between case one and two is under investigation to identify the possible advantages of one over the other. For this case the external acceleration including the gravity ($\hat{a}_e$) is assumed to act along the sense direction and appears as a bias force in the equation of motion. Therefore the external acceleration appears as an independent forcing term in the right hand side of (2) modifying the initial curvature of the beam and the electrostatic force.

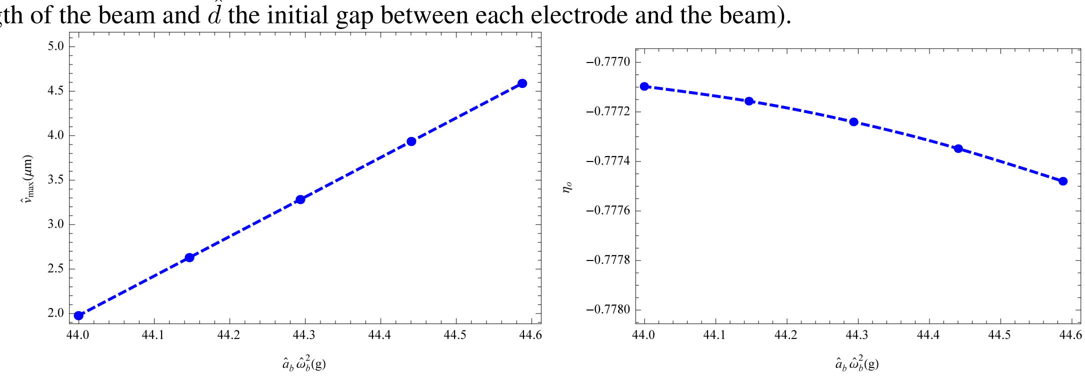
## Results

We perform a preliminary analysis to characterize the effect of quadratic damping varying the coefficient $\hat{\beta}$ from $0.04\text{kg/m}^2$ to $0.12\text{kg/m}^2$. The frequency-response curves and the maximum amplitude of the response vs. $\hat{\beta}$ are plotted in Figure 2a and Figure 2b, respectively. For operation in air we set $\hat{\beta} = 0.08\text{kg/m}^2$, for operation in medium vacuum $\hat{\beta} = 0.04\text{kg/m}^2$, and for operation in high vacuum $\hat{\beta} = 0.02\text{kg/m}^2$. At the extrema we have $\mathrm{d}\|v\|/\,\mathrm{d}\sigma = 0$ and therefore $\mathrm{d}\|a_{pf}\|/\,\mathrm{d}\sigma = 0$. Figures 3a and 3b show the variation in the maximum amplitude and the corresponding phase for an increasing set of base-accelerations. The amplitude and the phase represent the solutions of the amplitude- and phase-equation describing the response of the beam in polar coordinate.




(a) (b)

Figure 2: The frequency-response curves (a) and the maximum response curve (b) for varying the quadratic damping coefficient ($\hat{\beta}$) other parameters are $Q = 30$, $\hat{V}_{\text{DC}} = 10\text{V}$, $\hat{a}_b = 1.5\mu\text{m}$, $- \circ -$ $\hat{\beta} = 0.02\text{kg/m}^2$, $-\blacksquare-$ $\hat{\beta} = 0.04\text{kg/m}^2$, $- \bullet -$ $\hat{\beta} = 0.06\text{kg/m}^2$, $-\square-$ $\hat{\beta} = 0.08\text{kg/m}^2$, and $- \triangle -$ $\hat{\beta} = 0.1\text{kg/m}^2$ ($\hat{\beta} = \frac{3\pi\hat{m}}{8\hat{d}}\beta$ where $\hat{m}$ represents the mass per unit length of the beam and $\hat{d}$ the initial gap between each electrode and the beam).




(a) (b)

Figure 3: The maximum response curve (a) and the corresponding phase (b) for increasing the base-acceleration parameter for an in-air operation of the accelerometer $Q = 30$ (other parameters are $\hat{V}_{\text{DC}} = 10\text{V}$, $\hat{\beta} = 0.08\text{kg/m}^2$).

## CONCLUSIONS

Unlike directly excited oscillators, the resonant response of parametrically excited oscillators is available only in a limited and well-defined frequency domain where it appears as an instability in addition to the forced response. Once the activation threshold is crossed, the resonant response grows rapidly as a function of the excitation level and provides a tool to measure the desired parameter effectively.

## References

[1] Crespo da Silva, M., Glynn, C. (1978) Nonlinear flexural-flexural-torsional dynamics of inextensional beams I. equations of motion. *Journal of Structural Mechanics* **6(4)**:437-448.

[2] Lajimi, S. A. M., Heppler, G. R. (2013) Free vibration and buckling of cantilever beams under linearly varying axial load carrying an eccentric end rigid body. *Transactions of the Canadian Society for Mechanical Engineering*, **37 (1)**: 89-109.

[3] Anderson, T., Nayfeh, A., Balachandran, B. (1996) Experimental verification of the importance of the nonlinear curvature in the response of a cantilever beam. *Journal of Vibration and Acoustics* **118 (1)**:21-27.