Control of amplitude chimeras by time delay in dynamical networks

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Summary. We investigate the influence of time-delayed coupling in a ring network of non-locally coupled Stuart-Landau oscillators upon chimera states, i.e., space-time patterns with coexisting partially coherent and partially incoherent domains. We focus on amplitude chimeras which exhibit incoherent behavior with respect to the amplitude rather than the phase and are transient patterns, and show that their lifetime can be significantly enhanced by coupling delay. To characterize their transition to phase-lag synchronization (coherent traveling waves) and other coherent structures, we generalize the Kuramoto order parameter. Contrasting the results for instantaneous coupling with those for constant and for time-varying coupling delays, we demonstrate that the lifetime of amplitude chimera states and related partially incoherent states can be controlled, i.e., deliberately reduced or increased, depending upon the type of coupling delay.

A chimera state is a non-trivial partial synchronization state arising spontaneously in networks of identical oscillators [1, 2]. Its dynamics is characterized by a spatio-temporal pattern that consists of two or more spatially separated domains of synchronous (coherent) and asynchronous (incoherent) behavior, respectively. Chimera states have first been reported theoretically in a ring of phase oscillators with symmetric non-local coupling, where for special initial conditions the spatio-temporal domains of in-phase synchronized units coexist with desynchronized domains exhibiting spatially incoherent chaotic dynamics [3, 4].

In this paper, we investigate the influence of time delay on chimera states, specifically, the dynamical regimes and the lifetime of amplitude chimeras in a ring network of Stuart-Landau oscillators with nonlocal time-delayed coupling. Amplitude chimeras occur in networks with coupled phase and amplitude dynamics, and they are distinguished from phase chimeras and amplitude-mediated chimeras by the fact that coherence and incoherence occur only with respect to the amplitude of the oscillators while all the elements of the network oscillate periodically with the same frequency and correlated phase [5, 6], and they have been found in Stuart-Landau networks with coupling which breaks the rotational S^1 symmetry. In contrast to classical chimeras [7] for amplitude chimeras the transient time decreases and saturates for large system size [8].

We investigate non-locally coupled ring networks of N oscillators with different types of delay in the coupling introduced via the delay operator $\mathcal{D}(\cdot)$. The local dynamics of the nodes is described by the Stuart-Landau oscillator, i.e., the normal form of a supercritical Hopf bifurcation. The dynamical equations are given by:

$$\dot{z}_j = \left[\lambda + i\omega - |z_j|^2\right] z_j + \frac{\sigma}{2P} \sum_{k=j-P}^{j+P} \left[\mathrm{Re}(\mathcal{D}[z_k(t)]) - \mathrm{Re}(z_j(t))\right], \tag{1}$$

where $z_j = x_j + iy_j \in \mathbb{C}$, $\lambda, \omega \in \mathbb{R}$. The variable z_j describes the j-th node ($j = 1, 2 \ldots N$, all indices mod N), σ is the coupling strength, and P is the number of coupled neighbors in each direction on a ring. In polar coordinates, $z_j = r_j e^{i\theta_j}$, where $r_j = |z_j|$ and $\theta_j = \arg(z_j)$, the dynamics of the uncoupled system is given by $\dot{r}_j = (\lambda - r_j^2) r_j$ and $\dot{\theta}_j = \omega$. For $\lambda > 0$, the uncoupled delay-free system exhibits self-sustained limit cycle oscillations with radius $r_0 = \sqrt{\lambda}$ and frequency ω. Since in our simulations we fix $\omega = 2$, the period of oscillations is $T = \pi$, which we will also refer to as intrinsic period. Depending on the type of the delay in the coupling, the delay operator $\mathcal{D}$ could act upon the state function $z(t)$ in different forms, e.g., $\mathcal{D}_1[z(t)] = z(t - \tau)$ in the case of *constant* delay, $\mathcal{D}_2[z(t)] = z(t - \tau(t))$ for *time-varying* delay, with time dependence given by the function $\tau(t)$, or $\mathcal{D}_3[z(t)] = \int_{-\infty}^{t} G(t')z(t - t')\mathrm{d}t'$ for *distributed* delay, where $G(t)$ is a kernel characterizing the delay distribution. Here we consider coupling only in the real parts, since this breaks the rotational S^1 symmetry of the system which is a necessary condition for the existence of nontrivial steady states $z_j \neq 0$ and thus for oscillation death [9]. Therefore, the symmetry-breaking form of the interaction between the oscillators induces a set of inhomogeneous fixed points in addition to the homogeneous fixed point at the origin $r_j = 0$.

In the instantaneous coupling case $\mathcal{D}_0[z(t)] = z(t)$, the system Eq.(1) exhibits various dynamical regimes due to the interplay of the symmetry-breaking coupling between the individual oscillators and the nonlocal network topology, e.g., multi-cluster oscillation death [10]. In particular, this system demonstrates chimera behavior with respect to amplitude dynamics, i.e., amplitude chimeras [5, 6, 11]. Amplitude chimera states can appear as long transients, potentially lasting for hundreds or even thousands of oscillation periods before a coherent state is reached.

Another chimera pattern observed in system Eq.(1) shows chimera behavior of steady states and is called *chimera death* [5, 6, 11]. In this regime the oscillations are quenched in a peculiar way. The population of identical elements breaks up into two groups: (i) spatially coherent inhomogeneous steady state (oscillation death), where the neighboring nodes of the network are correlated forming a regular inhomogeneous steady state, and (ii) spatially incoherent oscillation death, where the sequence of populated branches of the inhomogeneous steady state of neighboring nodes is completely random.

Here we analyze the lifetime of amplitude chimeras and the impact of time-delayed coupling on the network dynamics. Including constant discrete time delay in the coupling changes both the dynamical regimes of the network and the lifetime of amplitude chimeras. The presence of delay in the coupling significantly prolongs the lifetime of amplitude chimeras. Time delay induces various partially incoherent states related to amplitude chimeras which are characterized by relatively long lifetimes. These spatio-temporal patterns include a typical symmetric amplitude chimera with two coherent domains of the same spatial width oscillating in antiphase, separated by spatially incoherent regions consisting typically of few oscillators.

Next, we investigate the influence of time-varying delay on the network dynamics. In particular, we choose a periodic deterministic modulation of the time-delay around a nominal (average) delay value τ_0 in the form of a sawtooth-wave modulation [12]:

$$\tau(t) = \tau_0 + \varepsilon \left[2 \left(\frac{\varpi t}{2\pi} \bmod 1 \right) - 1 \right], \tag{2}$$

and also in the form of a square-wave modulation:

$$\tau(t) = \tau_0 + \varepsilon \, \mathrm{sgn}[\sin(\varpi t)], \tag{3}$$

where ε and ϖ are the amplitude and the angular frequency of the corresponding delay modulations, respectively. In particular, for a sawtooth-wave modulation of the delay given by Eq. (2) we analyze the dynamical states of the network and the corresponding lifetimes of partially incoherent states. Note that the sawtooth-wave modulation of the delay does not have a significant influence on the various regimes if compared to the constant delay case. There is, however, an occasional appearance of partially incoherent states at small values of coupling strength σ and different values of the number of nearest neighbours P. They survive the simulation time, but the main region of amplitude chimeras around $\sigma = 13$ and small P is mostly unchanged with increasing modulation amplitude. The square-wave delay modulation is rather interesting, since in this case by increasing the modulation amplitude, the domains corresponding to partially incoherent states become drastically reduced, almost disappearing for larger values of modulation amplitude. The impact of the modulation of the coupling delay on the network dynamics in the square-wave case becomes more visible in the parameter plane of the coupling range P and the amplitude of delay modulation ε for constant coupling strength. Increasing the modulation amplitude ε results in a sequence of appearance and disappearance of the amplitude chimera regions. Such behavior is a characteristic feature of systems under square-wave delay modulation, and it has already been reported, for instance, in variable-delay feedback control with respect to the sequence of stability islands for successful fixed-point control [13].

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References

[1] M. J. Panaggio and D. M. Abrams: *Chimera states: Coexistence of coherence and incoherence in networks of coupled oscillators*, Nonlinearity **28**, R67 (2015).

[2] E. Schöll: *Synchronization patterns and chimera states in complex networks: interplay of topology and dynamics*, Eur. Phys. J. Spec. Top. **225**, 891 (2016), theme Issue on Mathematical Modeling of Complex Systems (ed. T. Bountis, A. Provata, G. Tsironis, J. Johnson).

[3] Y. Kuramoto and D. Battogtokh: *Coexistence of Coherence and Incoherence in Nonlocally Coupled Phase Oscillators.*, Nonlin. Phen. in Complex Sys. **5**, 380 (2002).

[4] D. M. Abrams and S. H. Strogatz: *Chimera states for coupled oscillators*, Phys. Rev. Lett. **93**, 174102 (2004).

[5] A. Zakharova, M. Kapeller, and E. Schöll: *Chimera death: Symmetry breaking in dynamical networks*, Phys. Rev. Lett. **112**, 154101 (2014).

[6] A. Zakharova, M. Kapeller, and E. Schöll: *Amplitude chimeras and chimera death in dynamical networks*, J. Phys. Conf. Series **727**, 012018 (2016).

[7] M. Wolfrum and O. E. Omel'chenko: *Chimera states are chaotic transients*, Phys. Rev. E **84**, 015201 (2011).

[8] S. Loos, J. C. Claussen, E. Schöll, and A. Zakharova: *Chimera patterns under the impact of noise*, Phys. Rev. E **93**, 012209 (2016).

[9] A. Zakharova, I. Schneider, Y. N. Kyrychko, K. B. Blyuss, A. Koseska, B. Fiedler, and E. Schöll: *Time delay control of symmetry-breaking primary and secondary oscillation death*, Europhys. Lett. **104**, 50004 (2013).

[10] I. Schneider, M. Kapeller, S. Loos, A. Zakharova, B. Fiedler, and E. Schöll: *Stable and transient multi-cluster oscillation death in nonlocally coupled networks*, Phys. Rev. E **92**, 052915 (2015).

[11] A. Zakharova, S. Loos, J. Siebert, A. Gjurchinovski, J. C. Claussen, and E. Schöll: *Controlling chimera patterns in networks: interplay of structure, noise, and delay*, in *Control of Self-Organizing Nonlinear Systems*, edited by E. Schöll, S. H. L. Klapp, and P. Hövel (Springer, Berlin, Heidelberg, 2016).

[12] A. Gjurchinovski, A. Zakharova, and E. Schöll: *Amplitude death in oscillator networks with variable-delay coupling*, Phys. Rev. E **89**, 032915 (2014).

[13] A. Gjurchinovski, T. Jüngling, V. Urumov, and E. Schöll: *Delayed feedback control of unstable steady states with high-frequency modulation of the delay*, Phys. Rev. E **88**, 032912 (2013).