Exact Model Reduction for a von Kármán beam

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Abstract

We apply the recently formulated technique of Slow-Fast Decomposition (SFD) towards the model order reduction of a von Kármán beam. SFD deals with the identification and calculation of slow manifold(s) in the underlying full system, which attracts nearby solution at rates faster than typical rates with the manifold, thereby allowing for a mathematically rigorous model reduction. This is a natural consquence of the geometric singular perturbation theory, applicable to special systems characterized by a dicotomy in time-scales. The beam is characterized by gemetrical nonlinearties and viscoelastic material damping and is an ideal candidate for application of SFD since it is characterized by the desirable stiff and flexible degrees of freedom in the axial and the transverse directions of motion, respectively.

1 Introduction

For general finite dimensional mechanical systems of the form

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} + \mathbf{F}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is a non-singular mass matrix, $\mathbf{F}(\mathbf{q},\dot{\mathbf{q}},t) \in \mathbb{R}^n$ is the vector containing the internal and external forces, Haller and Ponsioen [3] discuss the decomposition of the generalized displacement vector $\mathbf{q}$ as

$$\mathbf{q} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^{n_s}, \quad \mathbf{y} \in \mathbb{R}^{n_f}, \quad n_s + n_f = n,$$

where $\mathbf{x}$ and $\mathbf{y}$ label the potentially relatively flexible (*slow*) and stiff (*fast*) unknowns, respectively. This potentially slow/fast dependence is explicitly modeled by a small physical parameter ϵ in the system and considering $\mathbf{M}$ and $\mathbf{F}$ as smooth function of $\frac{\mathbf{y}}{\epsilon}$. The system (1) can then be written in the following inertially-decoupled form:

$$\mathbf{M}_1\left(\mathbf{x}, \frac{\mathbf{y}}{\epsilon}, t; \epsilon\right)\ddot{\mathbf{x}} - \mathbf{Q}_1\left(\mathbf{x}, \dot{\mathbf{x}}, \frac{\mathbf{y}}{\epsilon}, \dot{\mathbf{y}}, t; \epsilon\right) = \mathbf{0},$$
$$\mathbf{M}_2\left(\mathbf{x}, \frac{\mathbf{y}}{\epsilon}, t; \epsilon\right)\ddot{\mathbf{y}} - \mathbf{Q}_2\left(\mathbf{x}, \dot{\mathbf{x}}, \frac{\mathbf{y}}{\epsilon}, \dot{\mathbf{y}}, t; \epsilon\right) = \mathbf{0},$$

where $\mathbf{M}_1 \in \mathbb{R}^{n_s\times n_s}$, $\mathbf{M}_2 \in \mathbb{R}^{n_f\times n_f}$ are mass matrices which can be derived from $\mathbf{M}(\mathbf{q},t)$, and the terms $\mathbf{Q}_1 \in \mathbb{R}^{n_s}$, $\mathbf{Q}_2 \in \mathbb{R}^{n_f}$ are the forces on the slow and fast degrees of freedom respectively.

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Haller and Ponsioen [3] deduced conditions under which such a partition leads to an *exact* reduced-order model (as defined by Haller and Ponsioen [3]) of the system (1). After the introduction of mass-normalized forcing terms using a new variable $\boldsymbol{\eta} = \frac{\mathbf{y}}{\epsilon}$ as

$$\begin{aligned} \mathbf{P}_1(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, \tau; \epsilon) &= -\mathbf{M}_1^{-1}\mathbf{Q}_1\left(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, t; \epsilon\right) , \\ \mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, \tau; \epsilon) &= -\epsilon\mathbf{M}_2^{-1}\mathbf{Q}_1\left(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, t; \epsilon\right) , \end{aligned} \tag{2}$$

these conditions are given by:

(A1) Nonsingular extension to $\epsilon = 0$: Both $\mathbf{P}_1$ and $\mathbf{P}_2$ must possess smooth extension to their respective $\epsilon = 0$ limits.

(A2) Existence of a critical manifold: The algebraic equation $\mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \mathbf{0}, \tau; 0) \equiv \mathbf{0}$ can be solved for $\boldsymbol{\eta}$ in terms of $(\mathbf{x}, \dot{\mathbf{x}}, \tau)$ on an open bounded domain $\mathcal{D}_0 \subset \mathbb{R}^{n_s} \times \mathbb{R}^{n_s} \times \mathcal{T}$. Such a solution, denoted as $\boldsymbol{\eta} = \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau)$, represents the critical manifold.

(A3) Asymptotic stability of the critical manifold: The equilibrium solution $\boldsymbol{\eta} \equiv \mathbf{0} \in \mathbb{R}^{n_f}$ of the unforced, constant-coefficient linear system

$$\boldsymbol{\eta}'' + -\partial_{\dot{\mathbf{y}}}\mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \mathbf{0}, \tau; 0)\boldsymbol{\eta}' + -\partial_{\boldsymbol{\eta}}\mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \mathbf{0}, \tau; 0)\boldsymbol{\eta} = \mathbf{0}$$

is asymptotically stable for all fixed parameter values $(\mathbf{x}, \dot{\mathbf{x}}, \tau) \in \mathcal{D}_0$.

The main result of Haller and Ponsioen [3] establishes general expression for this reduced-order model for all $(\mathbf{x}, \dot{\mathbf{x}}, \tau) \in \mathcal{D}_0$ as

$$\ddot{\mathbf{x}} - \overline{\mathbf{P}_1} - \epsilon\left[\overline{\partial_{\boldsymbol{\eta}}\mathbf{P}_1}\mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \overline{\partial_{\dot{\mathbf{y}}}\mathbf{P}_1}\mathbf{H}_\mathbf{0}(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \overline{\partial_\epsilon\mathbf{P}_1}\right] + \mathcal{O}\left(\epsilon^2\right) = \mathbf{0},$$

which describes the reduced flow over a $2n_s$-dimensional invariant manifold (called the slow manifold) along which the position and velocities in the stiff unknowns ($\mathbf{y}$,$\dot{\mathbf{y}}$) can be expressed as a graph over the flexible ones ($\mathbf{x}$,$\dot{\mathbf{x}}$). Furthermore, if these conditions are satisfied, then the trajectories of the full system (close enough to the slow manifold in the phase space and ϵ small enough) synchronize with the reduced model trajectories at rates faster than those within the slow manifold.

2 SFD of von Kármán beam

Symbol	Description (unit)
L	Length of beam (m)
h	Height of beam (m)
E	Young's Modulus (Pa)
κ	Viscous damping rate of material (Pa s)
ρ	Density (kg/m^3)
τ	Non-dimentionalized time

A suitable non-dimensionalization of the Partial Differential Equations (PDEs) governing the behavior of a flat von Kármán beam (cf. Reddy [1]), along with assumptions of Kelvin-voigt for modelling material viscoelasticity and uniform, rectangular cross section, leads to the following non-dimensionalized PDEs:

$$\begin{aligned} \ddot{w} + \frac{1}{12}\partial_x^4 w + \frac{\zeta\epsilon}{12}\partial_x^4\dot{w} - \frac{1}{\epsilon}\partial_x\left(\partial_x u\,\partial_x w\right) - \zeta\partial_x(\partial_x\dot{u}\,\partial_x w) \\ -\frac{1}{2}\partial_x\left(\partial_x w\right)^3 - \zeta\epsilon\partial_x\left(\left(\partial_x w\right)^2\partial_x\dot{w}\right) = \alpha q(x,\tau)\,, \\ \ddot{u} - \frac{1}{\epsilon}\partial_x^2 u - \frac{\zeta}{\epsilon}\partial_x^2\dot{u} - \frac{1}{2\epsilon}\partial_x\left(\partial_x w\right)^2 - \zeta\partial_x\left(\partial_x w\partial_x\dot{w}\right) = \beta p(x,\tau)\,, \end{aligned} \tag{3}$$

where the beam is aligned along the x direction in the undeformed state, $u(x,\tau)$, $w(x,\tau)$ denote the displacements in the axial and transverse direction of the beam, respectively, $\epsilon = \frac{h}{L}$ is the beam thickness to length ratio, α, β are load scaling parameters, $\zeta = \frac{\kappa\rho^{1/2}}{E^{3/2}L}$ is a dimensionless constant. Upon Finite-Element discretization of the non-dimensional system (3) with cubic shape functions for w and linear shape functions for u (see e.g. Crisfield [2]), we obtain the finite-dimensional discretized version of (3) as

$$\mathbf{M}_1\ddot{\mathbf{x}} + \zeta\epsilon\left(\mathbf{K}_1 + \boldsymbol{\mathcal{C}}(\mathbf{x})\right)\dot{\mathbf{x}} + \zeta\boldsymbol{\mathcal{D}}(\mathbf{x})\dot{\mathbf{y}} + \mathbf{K}_1\mathbf{x} + \frac{1}{\epsilon}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{y}) + \boldsymbol{\mathcal{G}}(\mathbf{x}) = \alpha\mathbf{q}(\tau)\,,$$
$$\mathbf{M}_2\ddot{\mathbf{y}} + \frac{\zeta}{\epsilon}\mathbf{K}_2\dot{\mathbf{y}} + \zeta\boldsymbol{\mathcal{E}}(\mathbf{x})\dot{\mathbf{x}} + \frac{1}{\epsilon^2}\mathbf{K}_2\mathbf{y} + \frac{1}{\epsilon}\boldsymbol{\mathcal{H}}(\mathbf{x}) = \beta\mathbf{p}(\tau), \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^{n_s}, \mathbf{y} \in \mathbb{R}^{n_f}$ are the finite dimensional (discretized) counterparts of the unknowns w, u respectively (n_s, n_f being the number of unknowns dependent on the finite-element discretization), and $\mathbf{M}_1, \mathbf{K}_1 \in \mathbb{R}^{n_s \times n_s}$ and $\mathbf{M}_2, \mathbf{K}_2 \in \mathbb{R}^{n_f \times n_f}$ are the corresponding mass and stiffness matrices. Furthermore, $\boldsymbol{\mathcal{F}}$ (a bilinear function in $\mathbf{x}, \mathbf{y}$), $\boldsymbol{\mathcal{G}}$ (a cubic function in $\mathbf{x}$), $\boldsymbol{\mathcal{H}}$ (a quadratic function in $\mathbf{x}$) correspond to the nonlinear elastic force vectors in the beam, and $\boldsymbol{\mathcal{C}}$ (a quadratic functionin $\mathbf{x}$), $\boldsymbol{\mathcal{D}}$ (a linear function in $\mathbf{x}$), $\boldsymbol{\mathcal{E}}$ (a linear function in $\mathbf{x}$) correspond to the nonlinear damping matrix contributions.

It can be shown that (**A1**)-(**A3**) are satisfied for the system (4), and as deduced for general mechanical systems by Haller and Ponsioen [3], the system admits an exact reduced order model given by:

$$\mathbf{M}_1\ddot{\mathbf{x}} + \mathbf{K}_1\mathbf{x} + \boldsymbol{\mathcal{F}}\left(\mathbf{x}, \mathbf{G}_0(\mathbf{x},\dot{\mathbf{x}},\tau)\right) + \boldsymbol{\mathcal{G}}(\mathbf{x}) +$$
$$\epsilon\left[\underbrace{\overline{\partial_{\boldsymbol{\eta}}\boldsymbol{\mathcal{F}}\left(\mathbf{x},\boldsymbol{\eta}\right)}\mathbf{G}_1(\mathbf{x},\dot{\mathbf{x}},\tau)}_{\text{conservative correction}} + \underbrace{\zeta\left(\boldsymbol{\mathcal{D}}(\mathbf{x})\mathbf{H}_0(\mathbf{x},\dot{\mathbf{x}},\tau) + \left(\mathbf{K}_1 + \boldsymbol{\mathcal{C}}(\mathbf{x})\right)\dot{\mathbf{x}}\right)}_{\text{damping terms}}\right] + \mathcal{O}(\epsilon^2) = \alpha\mathbf{q}(\tau), \tag{5}$$

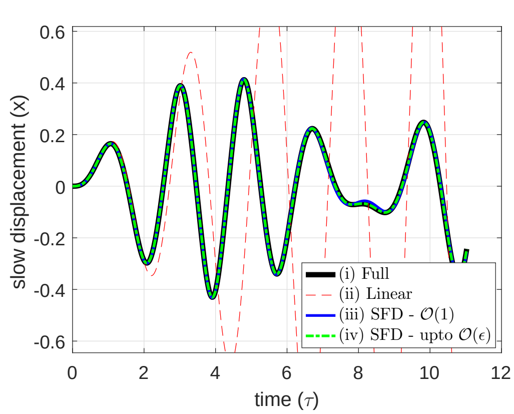
where

$$\mathbf{G}_0(\mathbf{x},\dot{\mathbf{x}},\tau) = -\mathbf{K}_2^{-1}\boldsymbol{\mathcal{H}}(\mathbf{x})\,,$$
$$\mathbf{H}_0(\mathbf{x},\dot{\mathbf{x}},\tau) = -\mathbf{K}_2^{-1}\left[\partial_{\mathbf{x}}\boldsymbol{\mathcal{H}}(\mathbf{x})\right]\dot{\mathbf{x}}\,,$$
$$\mathbf{G}_1(\mathbf{x},\dot{\mathbf{x}},\tau) = -\zeta\left(\mathbf{H}_0(\mathbf{x},\dot{\mathbf{x}},\tau) + \mathbf{K}_2^{-1}\boldsymbol{\mathcal{E}}(\mathbf{x})\dot{\mathbf{x}}\right) + \beta\mathbf{K}_2^{-1}\mathbf{p}(\tau)\,. \tag{6}$$

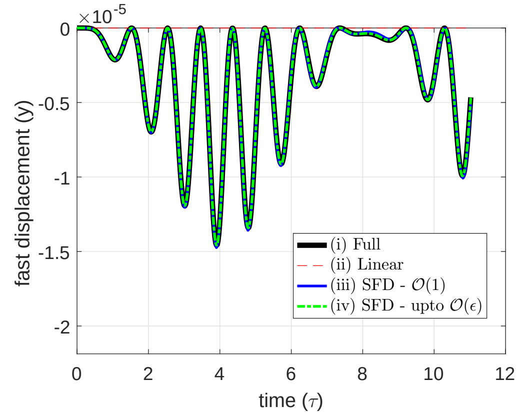
It is interesting to see that the reduced order model (5) is conservative (contains only inertial and elastic force terms) at the leading order, where as the full system (4) is characterized by viscoelastic damping. The $\mathcal{O}(\epsilon)$ terms in the reduced-order model contain these damping contributions, and hence are important from a physical point of view. Apart from the damping contributions, there also exists a conservative correction at the $\mathcal{O}(\epsilon)$ level which also includes the static response of $\mathbf{y}$ variables to the corresponding loading $\beta\mathbf{p}(\tau)$ (cf. expression for $\mathbf{G}_1(\mathbf{x},\dot{\mathbf{x}},\tau)$ in (6)).

3 Preliminary results

We consider a beam with geometrical and material parameters given by: length $L = 1$ m, thickness to length ratio ϵ in the range from $10^{-4} - 10^{-2}$, Young's Modulus $E = 70$ G Pa, density$\rho = 2700$ Kg/m^3, material viscous damping rate $\kappa = 10^8$ Pa s. We use a spatially uniform load on the beam in the axial as well as the transverse direction given by $\alpha = 1$, $\beta = 1$, $q(x,\tau) = p(x,\tau) = \sin\left(\Omega T_0\tau\right)$, where $T_0 = \frac{L}{\epsilon}\sqrt{\frac{\rho}{E}}$ is the constant used to non-dimensionalize time and Ω is the loading frequency (chosen to be the first natural frequency of the beam in this case). Using these parameters we obtain $\zeta \approx 7.2739$.

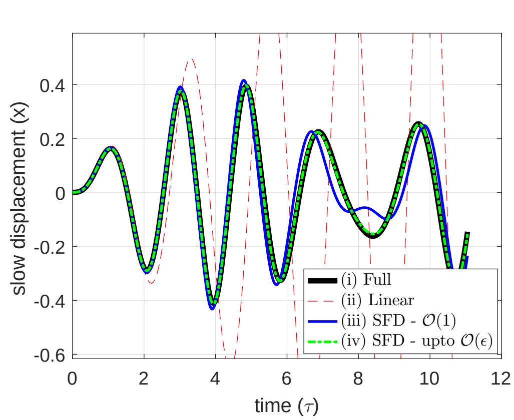


(a) Displacement at quarter length of beam in the transverse direction (Slow **x** DOF).

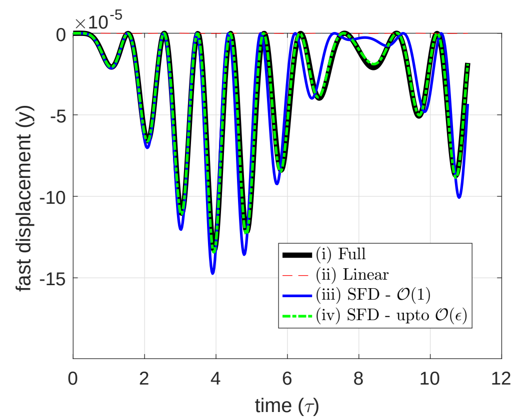


(b) Displacement at quarter length of beam in the axial direction (Fast **y** DOF).

Figure 1: Comparison of solution for slow and fast variables in reduced solution with their full nonlinear and linearized counterparts for $\epsilon = 10^{-4}$. Note that for such small values of ϵ the reduced model at the leading order (containing only conservative terms) is a good enough representation of the full system, and is practically identical to ROMs obtained by inclusion of the $\mathcal{O}(\epsilon)$ and $\mathcal{O}\left(\epsilon^2\right)$ terms.



(a) Displacement at quarter length of beam in the transverse direction (Slow **x** DOF).



(b) Displacement (in m) at quarter length of beam in the axial direction (Fast **y** DOF).

Figure 2: Comparison of solution for slow and fast variables in reduced solution with their full nonlinear and linearized counterparts for $\epsilon = 10^{-3}$. The reduced model at the leading order not accurate enough, $\mathcal{O}(\epsilon)$ terms (which include damping contributions) required to improve accuracy.

We observe that for ϵ of order 10^{-4}, the reduced-order at the leading order (which is conservative) provides a good approximation for the full system. The $\mathcal{O}(\epsilon)$ terms, however, become important when ϵ is of order 10^{-3} since they include the damping contributions which also become significant as ϵ increases. Further increase of ϵ to values higher than 10^{-2} leads to significant loss in accuracy of the reduced model. It is then interesting to compute the $\mathcal{O}\left(\epsilon^2\right)$ terms in the reduced model and check the effect on accuracy. The corresponding results are under process.

4 Current work and Conclusion

We constructed and tested a reduced-order model for the von-Karman beam using the slow-fast decomposition (SFD). This reduced-order model (5) has the advantage that the slow manifold is a *global* structure in the phase space which attracts trajectories off the slow manifold with rates faster that those within the slow manifold, as opposed to being local when the reduction is justified only in a neighbourhood of an equilibrium point in the phase space. Though the SFD reduction is robust and effective, the reduced system contains n_s unknowns, which can still be presumably large. Particularly in the current beam example, due to the chosen shape functions for discretization, it is easy to see that the reduced model (5) obtained from SFD would still contain two-third of the total number unknowns in the full system (4).

The eigenvalue analysis of the linearized SFD-reduced system shows the existence of a further separation in time scales, thus indicating a few linear modal subspaces around the equilibrium which are slower than the rest. The smoothest invariant manifold which is tangent to and local extension of a linear modal subspace is known as the Spectral Submanifold (SSM), as introduced by Haller and Ponsioen [4]. These SSMs corresponding to a slow subspace (i.e. the subspace spanned by eigenvectors corresponding to eigenvalues with lowest magnitude real parts) are expected to be useful in further reduction of a SFD-based reduced model (5) and is the focus of our current work. This is especially useful when the system exhibits slow and fast time scales, but SFD is not applicable either due to the conditions (A1)-(A3) not being satisfied, or a partition of unknowns of the mechanical system into slow and fast degrees of freedom not being intuitively clear.

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