Different models for balancing using accelerometer

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Summary. Stabilization of a pinned pendulum about its upright position by delayed state feedback is considered, where the pendulum's angular position is measured by a single accelerometer attached directly to the pendulum. It is shown that, by applying different simplifications in the mechanical model, the governing equation can be retarded functional differential equation, neutral functional differential equation and even advanced functional differential equation. The different models are compared by means of their stability properties.

Intorduction

Functional differential equations (FDEs) describe systems, where the rate of change of the state depends on the state at deviating arguments. Typically these equations can be categorized into three groups [1]. (1) If the rate of change of the state depends on the past state of the system, then the corresponding mathematical model is a *retarded functional differential equation (RFDE)*. (2) If the rate of change of the state depends on the past values of both the state and its rate of change then the governing equation is called *neutral functional differential equation (NFDE)*. (3) If the rate of change of the state depends on the past values of higher derivatives of the state then the system is described by *advanced functional differential equations (AFDEs)*. While stability properties of RFDEs and NFDEs depends in the system and control parameters, AFDEs are always unstable with infinitely many unstable characteristic roots. Therefore real physical and engineering problems are usually described by RFDEs or NFDEs [2]. In this paper, a balancing task is modeled using different concepts and resulting in a variety of governing equations involving RFDEs, NFDEs and AFDEs. The different models are compared by means of stability diagrams.

Mechanical models

A simple mechanical model of a balancing tasks is shown on Figure 1. The model is a pinned inverted pendulum controlled by a torque about the pin. Similar models are often used to analyze human postural sway [3]. It is assumed that the deviation of the pendulum from vertical is measured by a piezo-ceramic crystal accelerometer, which is modeled as spring-mass system. Thus, the model has two degrees of freedom, the angular position φ and the relative displacement ξ of the block of mass m_0 in the accelerometer. The linearized equations of motion reads

$$\left(J_{\mathrm{o}}+ a^2 m_0\right)\ddot{\varphi}(t) + a m_0 \ddot{\xi}(t) - \left(\frac{l}{2} g m - a g m_0\right)\varphi(t) - g m_0 \xi(t) = Q(t), \quad (1)$$

$$a m_0 \ddot{\varphi}(t) + m_0 \ddot{\xi}(t) - k_0 \dot{\xi}(t) - g m_0 \varphi(t) + s_0 \xi(t) = 0, \quad (2)$$

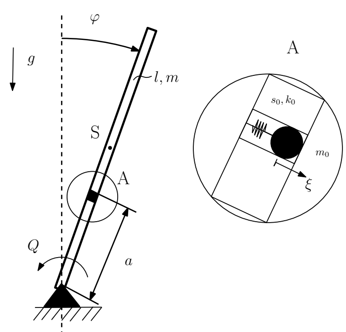

Figure 1: Mechanical model.

where l, m and J_{o} are the length, the mass and the mass moment of inertia with respect to the axis normal to the plane of the figure through point O of the pendulum, m_0, s_0 and k_0 are the mass, stiffness and damping in the model of the accelerometer, a is the distance between the suspension point O and the accelerometer and Q is the control torque. The control torque is assumed to be calculated using the displacement ξ, which, in case of a quasi-static model with small deviations, is proportional to the angular position φ. A PD feedback is assumed, thus $Q = p\,\xi + d\,\dot{\xi}$, where p and d are the proportional and the derivative control gains. Four concepts are considered for the calculation of the control force Q.

(C1) Real-time analogue control: $Q(t) = p\,\xi(t) + d\,\dot{\xi}(t)$.

(C2) Analogue control with feedback delay τ: $Q = p\,\xi(t-\tau) + d\,\dot{\xi}(t-\tau)$

(C3) Discrete-time control with sampling period h and with feedback delay: $Q(t) = p\,\xi(t_{j-r}) + d\,\dot{\xi}(t_{j-r})$ with $t \in [t_j, t_{j+1}]$, $t_j = jh$, $j \in \mathbb{Z}^+$. Here $r \in \mathbb{N}$ is the delay parameter and rh is the feedback delay.

(C4) Discrete-time control with numerically calculated derivative term: $Q(t) = p\,\xi(t_{j-r}) + d\,\frac{\xi(t_{j-r})-\xi(t_{j-r-1})}{h}$ with $t \in [t_j, t_{j+1}]$, $t_j = jh$, $j \in \mathbb{Z}^+$.

Related to the mechanical model of the accelerometer, we consider four different models.

(M1) In a quasi-static model, we assume that all the terms involving m_0 and k_0 are negligible except for the static term $g m_0 \varphi(t)$. In this case equation (1) can be simplified to

$$J_{\mathrm{o}}\ddot{\varphi}(t) - \frac{l}{2} g m \varphi(t) = Q(t) \quad (3)$$

with $\xi(t) = \frac{g m_0}{s_0}\varphi(t)$. This model result in a standard RFDE model of balancing (see [4]).

(M2) In a semi-quasi-static model, we assume that again all the terms involving m_0 and k_0 are negligible except for two terms, namely $gm_0\varphi(t)$ and $am_0\ddot{\varphi}(t)$. The corresponding equation of motion is the same as (3) with $\xi(t) = \frac{gm_0}{s_0}\varphi(t) - \frac{am_0}{s_0}\ddot{\varphi}(t)$. This mechanical model with control force model (C2) and (C3) results in an AFDE and in an NFDE, respectively. For more details on these models, see [4].

(M3) Another modeling concept is that only the term $m_0\ddot{\xi}(t)$ is neglected while the term $k_0\dot{\xi}(t)$ is not neglected. This model corresponds to a 1.5-degree-of-freedom model, since only the first derivative of the state variable $\xi(t)$ shows up in the governing equations.

(M4) No terms are neglected in (1).

Models M1 and M2 were analyzed in details in [4]. Here, we extend the analysis by models M3 and M4, which together with the four models of the control force give 8 different cases. Different models are referred as combination of the above notations. For instance, M1C1 refers to mechanical model M1 with control force model C1.

Results

The results are summarized in the form of stability diagrams for the different models in Figure 2. For the sake of completeness, the stability diagrams for models M2C1, M2C2, M2C3 and M2C4 are also presented. Models M3C1, M4C1 and M3C2, M4C2 give no stable region, i.e., the system with analogue controller cannot be stabilized independently whether delay is present in the feedback loop or not. The analogue controller with feedback delay (control force model C2) presents an interesting transition. Model M2C2 gives an AFDE for any $a > 0$ (see [4]). Model M4C2 gives an RFDE for any k_0 and a. The transition between these two models is provided by model M3C2. If $k_0 > 0$ and $a = 0$ then model M3C2 gives an RFDE, if $k_0 > 0$ and $a > 0$ then model M3C2 gives an NFDE and if $k_0 = 0$ and $a > 0$ then model M3C2 gives an AFDE. Depending on the size of the sampling period, the digitally control system can be stabilized, as the diagrams associated with models M3C3, M4C3 and M3C4, M4C4 show.

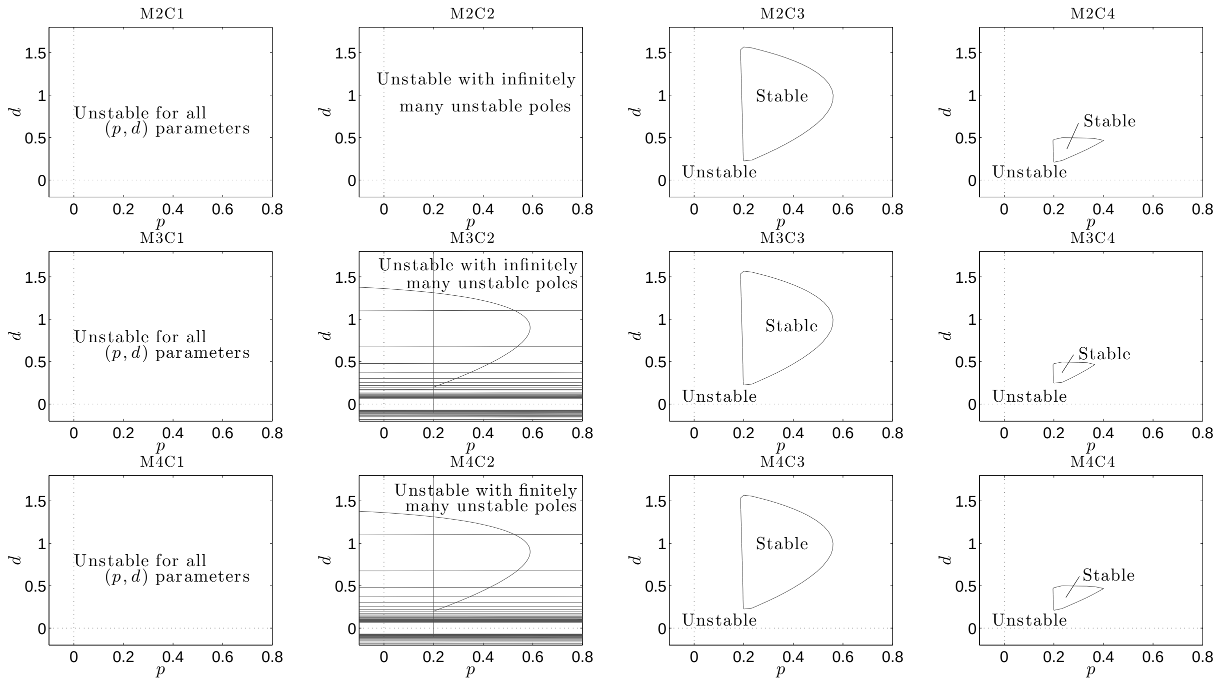

Figure 2: Stability diagrams for the different models.

References

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