Motion Planning Problem for a Finite-Dimensional Approximation of the Navier–Stokes Equations

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Summary. We consider a finite-dimensional projection of the Navier–Stokes equations describing the motion of a viscous incompressible fluid in a two-dimensional domain. This system is controlled by the forces corresponding to low-frequency modes. As an example, sufficient controllability conditions are proposed in terms of the first-order Lie brackets for the control-affine system under consideration. It is shown that the approximate motion planning problem is solvable by using a family of trigonometric control functions. An important feature of our control design scheme is that the control coefficients are computed explicitly in terms of solutions to auxiliary algebraic equations. We also discuss possible extensions of this approach for the case of controllability conditions with higher-order Lie brackets.

Derivation of the Equations of Motion

Consider the Navier–Stokes equations for the case of an incompressible fluid in a two-dimensional domain:

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)\,v + \frac{1}{\rho}\nabla p - \nu\Delta v = \sum_{j=1}^{m} u_j(t)F_j(x), \tag{1}$$

where $v=(v_1,v_2)$ is the velocity field, p is the pressure, ∇ is the gradient, Δ is the vector Laplace operator, $\nu\ge 0$ is the kinematic viscosity, $\rho>0$ is the density, and the dot stands for the inner product operation. The functions $p=p(t,x)$ and $v=v(t,x)$ depend on the time t and the spatial coordinates $x=(x_1,x_2)\in\mathbb{T}^2$. The right-hand side of system (1) describes the mass forces acting on the fluid. It is assumed that the action of controls $u_j(t)$ on the motion of the fluid is given in terms of functions $F_j(x)$, $j=1,2,...,m$. In addition to the Navier–Stokes system, we introduce the continuity equation:

$$\nabla\cdot v = 0. \tag{2}$$

The system of equations (1)–(2) is considered on the two-dimensional torus $x\in\mathbb{T}^2$, so that the functions $p(t,x)$ and $v(t,x)$ satisfy the periodic boundary conditions. By introducing the operator $\nabla^\perp$ and computing the vorticity $w=\nabla^\perp\cdot v=\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}$, we obtain the following system from the Navier–Stokes equations (1):

$$\frac{\partial w}{\partial t} + (v\cdot\nabla)w - \nu\Delta w = \sum_{j=1}^{m} u_j w_j(x), \tag{3}$$

where $w_j(x)=\nabla^\perp\cdot F_j(x)$. It is a well-known fact that each function $v(t,x)$ satisfying the continuity equation (2) may be uniquely reconstructed (up to a constant) from the function $w(t,x)$ [2]. We assume that the functions $v(t,x)$ and $w(t,x)$ have zero mean value in $\mathbb{T}^2$ for all t. Under these assumptions, the *controllability problem* for system (3) by applying a degenerate forcing on the torus has been studied in the paper [1]. In this work, we will construct a family of controls $u_j(t)$ in order to solve the *motion planning problem* for a finite-dimensional approximation of system (3).
Let us introduce the Fourier series for $w(t,x)$ with respect to the eigenfunctions $\{e^{ik\cdot x}\}$ of the Laplace operator on $\mathbb{T}^2$:

$$w(t,x)=\sum_{k\in\mathbb{Z}^2} q_k(t)e^{ik\cdot x},\quad q_0=0.$$

For any finite subset of indices $G\subset\mathbb{Z}^2$, we may consider the corresponding finite-dimensional projection of system (3) on the subspace spanned by $\{e^{ik\cdot x}|\ k\in G\}$, according to Galerkin's method (cf. [1]):

$$\dot q_k = \sum_{m+n=k}(m_1n_2-m_2n_1)|m|^{-2}q_mq_n - \nu|k|^2q_k + \sum_{j=1}^{m}u_j\phi_{jk},\quad k,m,n\in G, \tag{4}$$

where ϕ_{jk} are the Fourier coefficients of $w_j(x)$.
As an example, let us consider the Galerkin system for the following set of indices:

$$G=\{(k_1,k_2)\in\mathbb{Z}^2|\ |k_1|\le 1,\ |k_2|\le 1\}.$$

Note that $\bar q_k = q_{-k}\in\mathbb{C}$ as $w(t,x)$ is a real function, and $q_{0,0}\equiv 0$ due to the zero-mean assumption. We use the following notations for complex variables:

$$q_{1,1}=\xi_1+i\xi_2,\ q_{1,-1}=\xi_3+i\xi_4,\ q_{1,0}=\xi_5+i\xi_6,\ q_{0,1}=\xi_7+i\xi_8.$$

Then system (4) may be written in real coordinates as

$$\dot{\xi} = f_0(\xi) + \sum_{j=1}^{m} u_j f_j(\xi), \quad \xi = (\xi_1, \xi_2, ..., \xi_8)^T \in \mathbb{R}^8, \tag{5}$$

$$f_0(\xi) = -\nu\xi + \frac{1}{2}\begin{pmatrix} -2\nu\xi_1 \\ -2\nu\xi_2 \\ -2\nu\xi_3 \\ -2\nu\xi_4 \\ \xi_1\xi_7 + \xi_2\xi_8 - \xi_3\xi_7 + \xi_4\xi_8 \\ \xi_2\xi_7 - \xi_1\xi_8 - \xi_4\xi_7 - \xi_3\xi_8 \\ \xi_3\xi_5 + \xi_4\xi_6 - \xi_1\xi_5 - \xi_2\xi_6 \\ \xi_3\xi_6 - \xi_4\xi_5 - \xi_2\xi_5 + \xi_1\xi_6 \end{pmatrix}, \quad \nu \geq 0. \tag{6}$$

The components of $f_j(\xi) = (f_{j1}, f_{j2}, ..., f_{j8})^T$ are constants which can be represented in terms of the parameters ϕ_{jk} of system (4). Consider a particular case $m = 4$ and assume that the controls act on the chosen low-frequency modes as follows:

$$f_1 = \begin{pmatrix} f_{11} \\ 0 \\ 0 \\ 0 \\ f_{15} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 \\ f_{22} \\ 0 \\ 0 \\ 0 \\ f_{26} \\ 0 \\ 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 \\ 0 \\ f_{33} \\ 0 \\ 0 \\ 0 \\ f_{37} \\ 0 \end{pmatrix}, \ f_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_{44} \\ 0 \\ 0 \\ 0 \\ f_{48} \end{pmatrix}. \tag{7}$$

Motion Planning Problem

Following the approach of [3, 4], we introduce the sets of indices $S_0, S_1 \subseteq \{1, 2, ..., m\}$, $S_2 \subseteq \{1, 2, ..., m\}^2$, and consider the following family of controls:

$$u_k(t) = \sum_{i\in S_0} \delta_{ki} v_i + \sum_{i\in S_1} \delta_{ki} a_i \sin\left(\frac{2\pi K_i t}{\tau}\right) + \sum_{(i,j)\in S_2} a_{ij}\left\{\delta_{ki}\cos\left(\frac{2\pi K_{ij} t}{\tau}\right) + \delta_{kj}\sin\left(\frac{2\pi K_{ij} t}{\tau}\right)\right\}, \ k = 1, 2, ..., m, \tag{8}$$

where $\tau > 0$, v_i, a_i, a_{ij} are real parameters, K_i and K_{ij} are nonzero integers, and δ_{ki} is the Kronecker delta. The main result of this work concerns the solvability of the *approximate motion planning problem* for system (5) at time $\tau > 0$: given $\xi^0 \in \mathbb{R}^8$, $\xi^1 \in \mathbb{R}^8$, and $\epsilon > 0$, the goal is to construct an admissible control $u : [0, \tau] \to \mathbb{R}^m$ such that the corresponding solution $\xi(t; \xi^0, u)$ of system (5) satisfies the conditions $\xi(0; \xi^0, u) = \xi^0$ and $\|\xi(\tau; \xi^0, u) - \xi^1\| < \epsilon$. The above problem has a solution for *arbitrary* $\xi^0, \xi^1 \in \mathbb{R}^8$ and $\epsilon > 0$ only if system (5) is approximately controllable. Let us formulate sufficient controllability conditions.

Proposition 1. *Let $m = 4$, $\nu > 0$, $f_{jj} \neq 0$, and $f_{j,j+4} \neq 0$ for each $j = 1, 2, 3, 4$. Then the vector fields $f_i(\xi)$ satisfy the rank condition*

$$\operatorname{span}\left(f_i(\xi), [f_0, f_j](\xi) \,|\, i, j = 1, 2, 3, 4\right) = \mathbb{R}^8$$

in a neighborhood of $\xi = 0$, and the control-affine system (5) *with the vector fields given by* (6), (7) *is locally controllable at $\xi = 0$. Here $[f_0, f_j](\xi)$ denotes the Lie bracket.*

Conclusions

By exploiting Theorems 3.1 and 3.2 from the paper [4], we show that the approximate motion planning problem is solvable for system (5) with controls of the form (8) provided that the conditions of Propositon 1 hold, and $\|\xi^0\|$ and $\|\xi^0 - \xi^1\|$ are small enough. An important feature of our control design scheme is that the coefficients of (8) are computed in terms of solutions to auxiliary algebraic equations. As a possible direction for further study, we discuss an extension of this approach for the case of controllability conditions with higher-order Lie brackets.

References

[1] Agrachev A. A., Sarychev A. V. (2005) Navier–Stokes Equations: Controllability by means of Low Modes Forcing. *Journal of Mathematical Fluid Mechanics* **7**: 108-152.

[2] Gallavotti G. (2002) Foundations of Fluid Mechanics. Springer-Verlag, Berlin.

[3] Zuyev A. (2016) Exponential Stabilization of Nonholonomic Systems by Means of Oscillating Controls. *SIAM Journal on Control and Optimization* **54**: 1678-1696.

[4] Zuyev A., Grushkovskaya V. (2016) Motion Planning for Control-Affine Systems Satisfying Low-Order Controllability Conditions. *International Journal of Control* DOI: 10.1080/00207179.2016.1257157.