Response Regimes in Equivalent Mechanical Model of Weakly Nonlinear Liquid Sloshing

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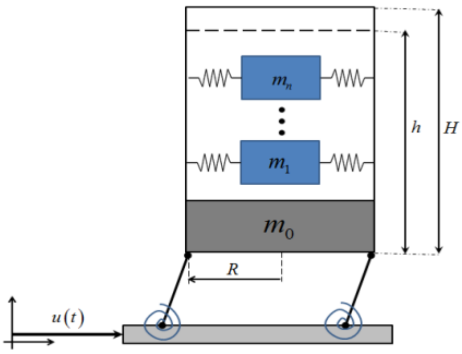
Summary. The current research considers nonstationary responses in reduced-order model (ROM) of partially liquid-filled tank under external forcing. The model involves one common degree of freedom for the tank and a non-sloshing liquid, and the other one – for the sloshing portion of the liquid. The coupling between these degrees of freedom is nonlinear, with the lowest-order potential allowed by symmetry considerations. Since the mass of the sloshing liquid in realistic conditions does not exceed 10% of the total mass of the system, the reduced-order model turns to be formally equivalent to well-studied oscillatory systems with nonlinear energy sinks (NES). Exploiting this analogy, and applying the methodology known from the studies of the systems with NES, we predict a multitude of possible nonstationary responses in the considered ROM. These responses conform, at least on the qualitative level, to the responses observed in experimental sloshing settings, multi-modal theoretical models and full-scale numeric simulations.

Introduction

Partially filled liquid storage tanks of different shapes are used in many engineering fields, including vehicles, sea crafts, aircraft, for the storage of various, maybe hazardous, liquids. The term “sloshing” refers to oscillatory relative motion of the liquid with respect to the containing vessel. The liquid sloshing may be hazardous for the vessel safety, since dynamic loads related to the sloshing may have direct and rather strong calamitous effect on the vessel stability and robustness. So far, elaborated analytical studies are limited to small-amplitude sloshing in rectangular and cylindrical vessels. Thus, efficient numerical and analytic tools are desired for assessing the coupling between the fluid and tank motions. While being most interesting and potentially hazardous, high-amplitude liquid sloshing in partially filled vessels still lacks thorough analytic representation. The sloshing liquid has infinite number of degrees of freedom; boundary conditions on the free surface are nonlinear and time-dependent. Near resonance, the nonlinear dynamical features may take place, for example multiple periodic solutions ('jump' phenomenon) [1], weakly quasi-periodic [2], [3] and strongly modulated responses [4]. Based on experimental results, it was pointed that cubic spring seems to describe the dynamical regimes in the best way. The nonlinear high-amplitude oscillations are modeled by addition of a cubic spring to the linear stiffness, describing both linear small-amplitude and nonlinear high-amplitude sloshing regimes. Conditions for existence and coexistence of periodic steady-state, weak-quasiperiodic and strongly modulated response (SMR) are obtained. The slow invariant manifold (SIM) describing the system slow flow-dynamics of 1:1 internal resonance is derived by multiple-scale analysis of the system. Finally, the results are compared qualitatively with previous experimental and computational data and show good agreement in terms of dynamical regimes. It is noteworthy that the framework of the presented asymptotic analysis is not limited to a specific tank shape. Besides, we assume infinite roof height in order to eliminate both liquid spilling and interaction between the fluid and the tank roof.

Model description and governing equations

Mechanical models of liquid sloshing often use an infinite series of pendula or mass-spring systems to represent the free liquid surface oscillation using infinite series of sloshing modes. These models, primarily developed using linear sloshing theory, are developed for various types of tank geometries and excitations. As shown by Abramson [1], the mass of each modal pendulum decreases rapidly with increasing mode number (figure 1(b)).

a) 

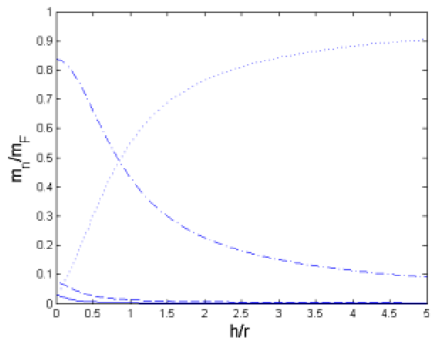
b) 

Figure 1-a)Scheme of multiple sloshing modes in partially-filled tank represented by a series of mass-spring systems; b) Ratios of the first three asymmetric sloshing modal masses and fixed mass m_0 to the total fluid mass m_f for cylindrical vessel; dotted-line, dashed-dotted-line ,dashed-line, solid-line, respectively.

Consequently, it is reasonable to take into account only the first mode in the mechanical equivalent model, as long as the excitation frequency is far from natural frequencies of the higher modes. The internal particle is located inside a straight cavity in the primary structure, and in contrast to earlier explored cubic NES [5], in addition to the cubic attachment represented by spring k, it is attached also through a linear viscosity c and linear spring with stiffness k_1 that represents the liquid first sloshing mode mass *m*. The linear spring is required to mimic the small-amplitude oscillatory sloshing motion due to gravity. By this addition we expect to observe several corrections to the results for the cubic NES. The cubic coupling is required to mimic moderated and high-amplitude sloshing, that according to earlier studies involve nonlinear characteristics associated with the cubic term. The external forcing is considered to be harmonic, with frequency Ω and amplitude $\bar{A}$.

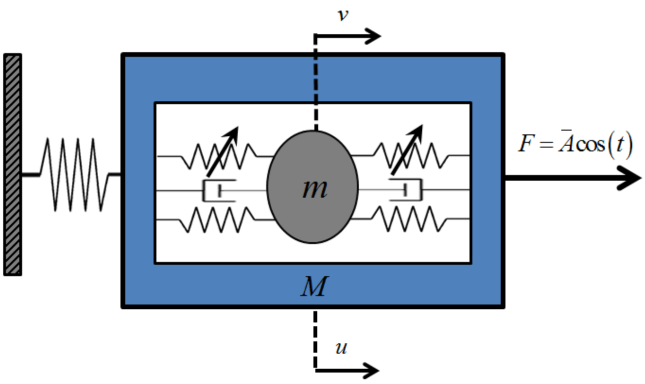


Figure 2-System scheme- linear oscillator as the primary system and internal particle with both linear and cubic attachments. The displacements of the primary mass and the impacting mass are denoted as u and v, respectively.

Hardening nonlinearity, corresponds to fluid level of $h < h^*$ and positive value of parameter *k*, whereas softening nonlinearity corresponds to $h < h^*$ and negative value of parameter *k*. By using the extended Hamilton's principle, we apply simple manipulations and time normalization $\tau = \Omega t$ to yield the following non-dimensional equations of motion:

$$\begin{aligned} &\ddot{u} + (1+\varepsilon\sigma)u + \varepsilon^2\beta(u-v) + \varepsilon\lambda(\dot{u}-\dot{v}) + \varepsilon k(u-v)^3 = \varepsilon A\cos(\tau) \\ &\ddot{v} + \varepsilon\beta(v-u) + \lambda(\dot{v}-\dot{u}) + k(v-u)^3 = 0 \end{aligned} \tag{1}$$

Here dots represent differentiation with respect to the normalized time τ, and the non-dimensional parameters governing the system dynamics are as follows:

$$\begin{aligned} &\omega_1^2 = \frac{k_1}{M}, \omega_2^2 = \frac{k_2}{m}, \Omega = \frac{\omega_1}{\sqrt{1+\varepsilon\sigma}}, \varepsilon A = \frac{\bar{A}}{M\Omega^2} \\ &\varepsilon\beta = \left(\frac{\omega_2}{\omega_1}\right)^2(1+\varepsilon\sigma), k = \frac{\bar{k}}{m\Omega^2}, \lambda = \frac{c}{m\Omega} \end{aligned} \tag{2}$$

The following coordinate transformation is used:

$$\begin{aligned} &X(t) = u(t) + \varepsilon v(t) \\ &w(t) = u(t) - v(t) \end{aligned} \tag{3}$$

Here w is the relative displacement of the internal particle with respect to the primary structure, and coordinate X is proportional to the displacement of the center-of-mass. The transformed non-dimensional equations of motion:

$$\begin{aligned} &\ddot{X} + \frac{1+\varepsilon\sigma}{1+\varepsilon}X + \frac{\varepsilon(1+\varepsilon\sigma)}{1+\varepsilon}w = \varepsilon A\cos(t) \\ &\ddot{w} + \frac{1+\varepsilon\sigma}{1+\varepsilon}X + \varepsilon\eta w + (1+\varepsilon)\lambda\dot{w} + k(1+\varepsilon)w^3 = \varepsilon A\cos(t) \end{aligned} \tag{4}$$

where $\eta = \frac{1+\varepsilon\sigma}{1+\varepsilon} + \beta(1+\varepsilon)$. In the following section, weakly nonlinear slashing periodic regimes will be described analytically.

Asymptotic description of dynamical regimes

We refer to two types of quasiperiodic responses; the first one is weak quasiperiodic response, which corresponds to supercritical loss of stability of the periodic regime and limit cycle formation through Hopf bifurcation. The second quasiperiodic regime, referred to as strongly modulated response (SMR), is associated with relaxation oscillations which are characterized by alternating fast and slow response amplitude variations. Detailed investigation of those regimes is described by Starosvetsky and Gendelman [5]. Asymptotic analysis of the resonance regimes mentioned above is achieved by complexification-averaging and multiple scales methods.

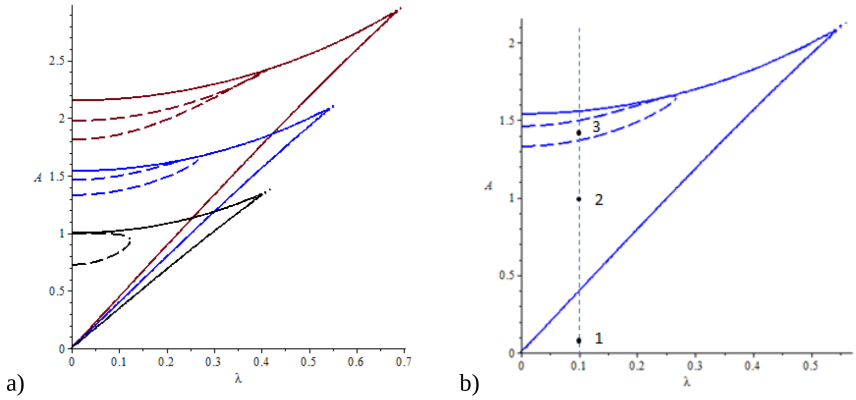


Figure 3- Projection of the saddle-node and Hopf bifurcations for: $k=4/3, \sigma=5, \varepsilon=0.05$*; SN bifurcations: solid line, Hopf bifurcation: dashed line. a) Red:* $\beta=0$*, blue:* $\beta=5$*, black:* $\beta=10$ *(color online); b) for* $\beta=5$*, point 1:* $A=0.1$ *existence of periodic solutions; point2:* $A=1$ *coexistence of both weakly and strongly quasiperiodic regimes; point 3:* $A=1.4$ *coexistence of both periodic and strongly quasiperiodic regimes.*

As one can see in Figure 3(b), the bifurcations locus consists of three different zones: point 1: existence of periodic solutions; point2: coexistence of both weakly and strongly quasiperiodic regimes; and point 3: coexistence of both periodic and strongly quasiperiodic regimes. If we take A as a bifurcation variable and λ as a parameter, we can calculate the critical maximum value of λ for SN bifurcation. The same method with respect to λ yields the critical value of A:

$$\lambda_{SN,cr} = \frac{1}{\sqrt{3}}\left|\frac{\sigma}{1-\sigma}+\varepsilon\beta\right|, \quad A_{SN,cr} = \frac{4}{9}\sqrt{\frac{2}{\left|k(\sigma-1)\right|}}\left(\sigma-\varepsilon\beta(\sigma-1)\right)^{3/2} \tag{5}$$

Critical damping value for existence of Hopf bifurcation $\lambda_{H,cr}$ is obtained by equating both of them:

$$\lambda_{H,cr} = \frac{1}{\sqrt{3}}\left|\bar{\beta}+\frac{\varepsilon\sigma-1}{(1+\varepsilon)}\right| \tag{6}$$

Hopf and SN bifurcations diagrams for fixed values of parameters corresponding to the three points are presented in Figure 4 (a-c).

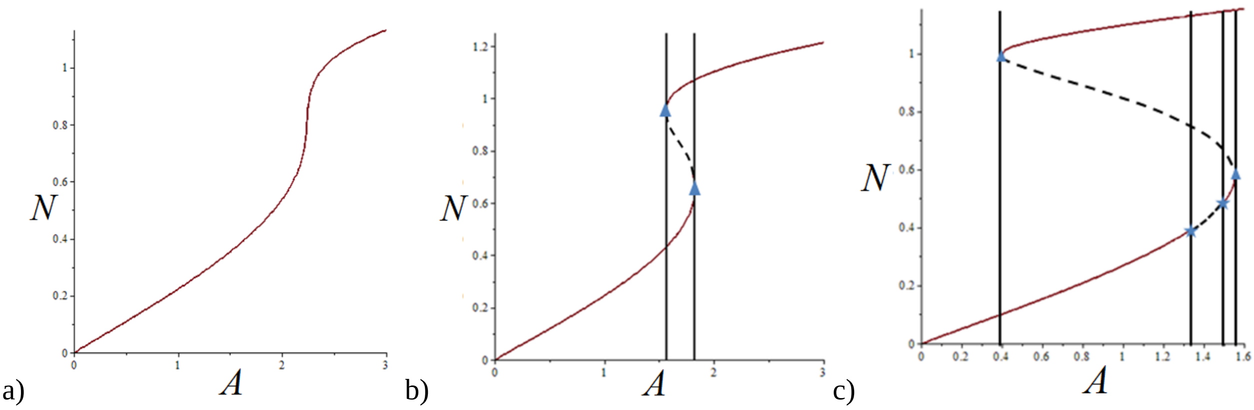


Figure 4- SN bifurcations (marked by traingles) and Hopf bifurcations (marked by stars) diagram for: $k=4/3, \sigma=5, \varepsilon=0.05$, $\beta=5 \Rightarrow \lambda_{cr,H}=0.268,\ \lambda_{cr,SN}=0.577$*. Stable solutions- solid lines, unstable solutions- dashed line; a)* $\lambda=0.6>\lambda_{cr,SN}$*; b)* $\lambda_{cr,H}<\lambda=0.4<\lambda_{cr,SN}$*; c)* $\lambda=0.1<\lambda_{cr,H}$.

Numerical results

In the following section, we compare the analytical predictions of the modulation envelope of the averaged system with numerical simulations of the full equations of motion (4). Following Figure 3(b), we examine the case that corresponds to the following parameter set: $k = 4/3, \sigma = 5, \varepsilon = 0.05, \beta = 5$ for different values of A and λ. The case of single periodic solution, corresponding to point 1 in Figure 3(b) and Figure 4, is shown in Figure 4 in terms of time history simulation and slow flow evolution on the SIM. The case of multiple periodic solution (amplitude 'jump' phenomenon), corresponding to point 2 in Figure 3(b), is shown in Figure 5. The case of coexistence of both periodic solution and weakly quasiperiodic regime, corresponding to point 3 in Figure 3(c), is shown in Figure 6:

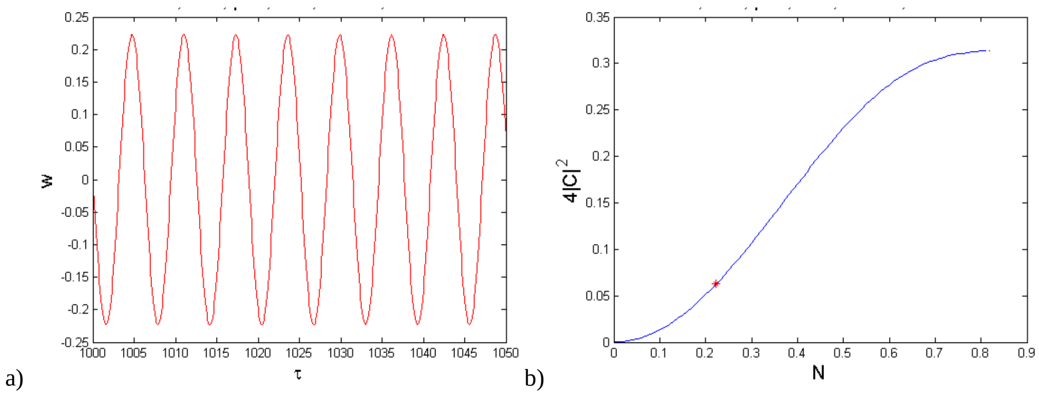


Figure 4- single periodic solution, corresponding to Figure 4(a): $k = 4/3, \sigma = 5, \varepsilon = 0.05$, $A = 1$ $\beta = 5 \Rightarrow \lambda_{cr,H} = 0.268,\ \lambda_{cr,SN} = 0.577$, $\lambda = 0.6 > \lambda_{cr,SN}$. *Initial conditions:* $u_0 = v_0 = \dot{u}_0 = \dot{v}_0 = 0$

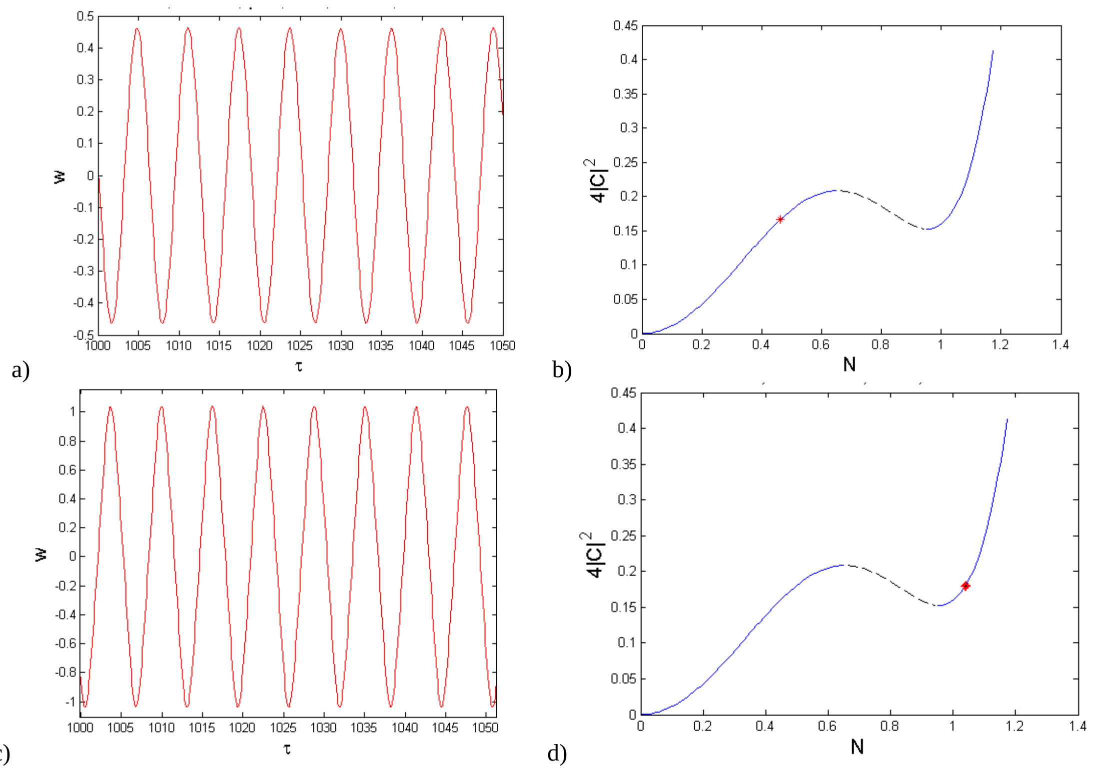
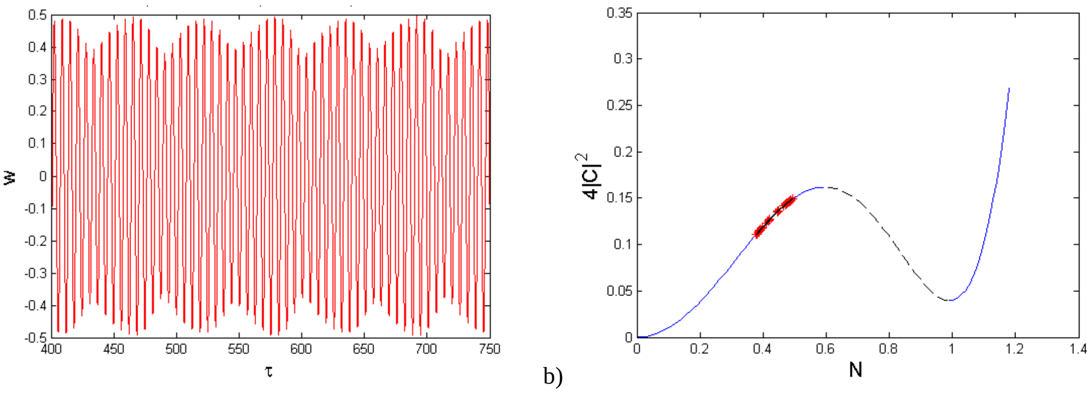


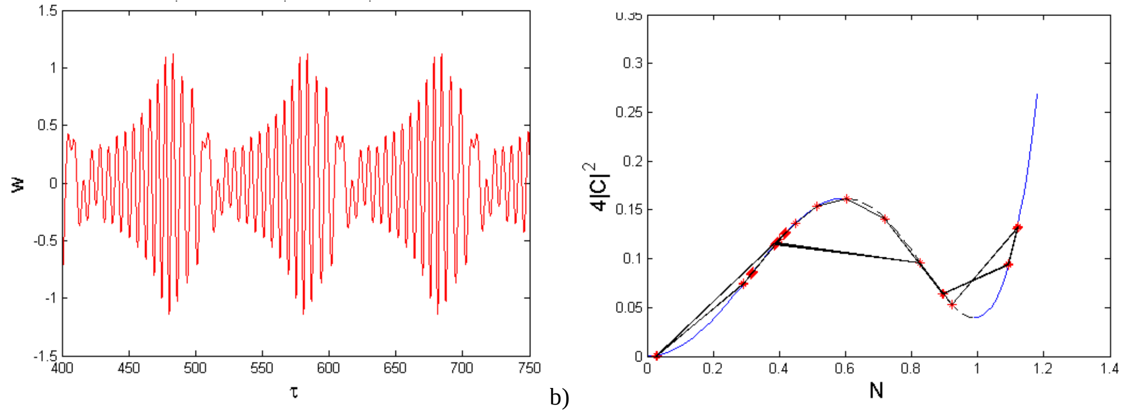
Figure 5-double periodic solution, corresponding to Figure 4(b): $k = 4/3, \sigma = 5, \varepsilon = 0.05$, $A = 1.62$, $\beta = 5 \Rightarrow \lambda_{cr,H} = 0.268,\ \lambda_{cr,SN} = 0.577$, $\lambda_{cr,H} < \lambda = 0.4 < \lambda_{cr,SN}$.

Initial conditions: a) $u_0 = v_0 = \dot{u}_0 = \dot{v}_0 = 0$, *b)* $u_0 = v_0 = 0\ \dot{u}_0 = 0.5\ \dot{v}_0 = 0$.



a) b)

Figure 6- Quasi-periodic solution, corresponding to Figure 4(c): $k = 4/3, \sigma = 5, \varepsilon = 0.05$, $A = 1.4$, $\beta = 5 \Rightarrow \lambda_{cr,H} = 0.268,\ \lambda_{cr,SN} = 0.577$, $\lambda = 0.1 < \lambda_{cr,H}$. *Initial conditions:* $u_0 = v_0 = 0$, $\dot{u}_0 = 0.5$, $\dot{v}_0 = 0$.

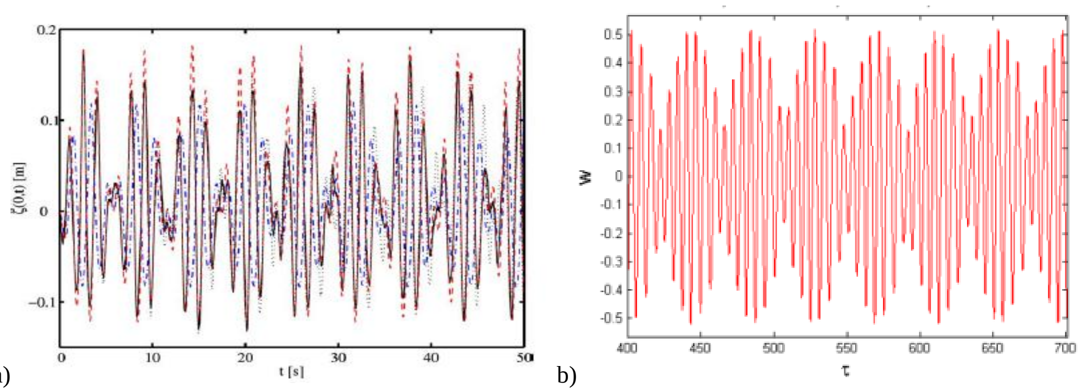


a) b)

Figure 7- SMR regime: $k = 4/3, \sigma = 0, \varepsilon = 0.05, \beta = 1$, $A = 0.3$, $\lambda = 0.2 \Rightarrow A > A_{cr,SMR} = 0.176$. *Initial conditions:* $u_0 = 0.29\ v_0 = -0.5$, $\dot{u}_0 = 0.9\ \dot{v}_0 = -0.15$.

Qualitative comparison with computational and experimental results

As mentioned in the introduction, many experimental and computational studies were performed in the field of weakly-nonlinear response regimes analysis of liquid sloshing in partially-filled tank subjected to horizontal periodic ground excitation. The vast majority of the sloshing regimes documented are two dimensional. The dynamical regimes revealed in previous sections are compared qualitatively with several experimental and computational studies. As one can observe in Figure 3, the comparison show qualitative similarity between the weakly-nonlinear dynamical regimes revealed with the help of the relatively simple reduced-order model introduced in the current study and the documented results.



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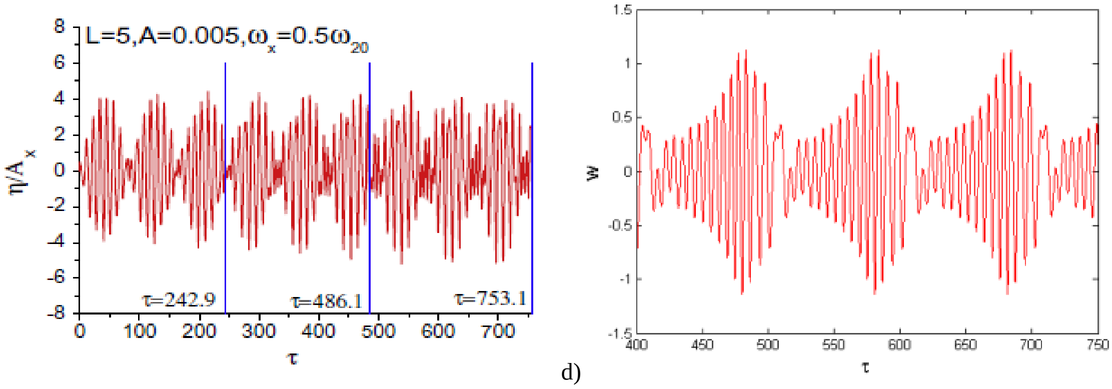

c) d)

Figure 8- Nonlinear beatings response regime in partially-filled rectangular tank subjected to horizontal periodic excitation. Free-surface elevation at the left wall in horizontally excited tank ; a) Faltinsen [2] *;b) Hill* [65]*; c) comparison between Frandsen and Faltinsen [3]; d)current model for parameter set:* $\lambda = 0.4, A = 0.3, \sigma = -0.2, \varepsilon = 0.1, k = 5$*;c) Strongly modulated response reigime (SMR); c)Zhang et al.* [4]*; d) Current model for parameter set:* $\lambda = 0.2, A = 0.3, \beta = 1, \sigma = 0, \varepsilon = 0.01, k = 4/3$.

Conclusions

Novel relatively simple two dimensional design of equivalent mechanical model was suggested for qualitatively describing and understanding the different nonlinear sloshing regimes and their leading mechanisms. The model uses well studied cubic NES systems methods to describe non-planar and weakly nonlinear sloshing regimes inside a partially filled vessel subjected to horizontal ground motion. The analytical predictions were validated numerically. Finally, qualitative comparison shows agreement between the regimes detected by this model and the regimes documented experimentally and numerically. This mechanical model paves the way towards a better stress assessment method for different engineering purposes.

References

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