Mixed synchronization in a triplet of coupled mechanical oscillators

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Summary. In this paper, the onset of mixed synchronization in a triplet of mechanical oscillators interacting via Huygens' coupling, i.e. a suspended rigid bar, is investigated. The term mixed synchronization refers to the case where two oscillators synchronize in-phase, while the third oscillator synchronizes in anti-phase with respect to the other two oscillators. Sufficient conditions for the onset of mixed synchronization are derived by using the Poincaré method and the obtained analytic results are complemented with numerical simulations. Ultimately, it is demonstrated that two synchronous modes can be observed in the coupled system at the same time namely, in-phase and anti-phase synchronization.

Introduction

Oscillations are common almost everywhere, be it in biology, in engineering, in economics, in physics or in other fields. It has been found that when two or more oscillatory systems are allowed to interact through a common medium, the systems may find their rhythms adjusted, i.e. they may synchronize [1]. Christiaan Huygens called this phenomenon in a more poetic way, he described the synchronization of two pendulum clocks as the *sympathy* of two clocks. Despite the lack of good modeling tools, Huygens did realize that there is a medium, called coupling, responsible for the onset of synchronized motion, namely the bar to which both pendula are attached. This medium is referred to as *Huygens' coupling* [2, 3].

In this work, a generalized version of the classical Huygens' experiment, which consists of three self-sustained oscillators mounted on a suspended rigid bar, is considered. It is demonstrated that the system exhibits mixed synchronization, i.e. two types of synchronization are observed simultaneously: two oscillators synchronize in-phase, whereas one oscillator synchronizes in anti-phase with respect to them. Under some mild assumptions, it is possible to obtain analytic conditions for the onset of synchronization. Specifically, conditions for the existence of mixed synchronous solutions are derived and likewise, the amplitude and frequency of these solutions are determined.

Modelling of the system

Consider the schematic model depicted in Figure 2. It consists of three actuated mass-spring damper oscillators mounted on a movable cart, which is attached to the fixed world by means of a spring and a damper. Using Newton's 2nd law, it follows that the idealized—i.e. assuming no friction is present —equations of motion of the system of Figure 2 are

$$\ddot{x}_i = -\omega^2(x_i - z) - 2\zeta\omega(\dot{x}_i - \dot{z}) + U_i, \quad i = 1, 2, 3, \tag{1}$$

$$\ddot{z} = -\omega_z^2 z - 2\zeta_z\omega_z\dot{z} - \mu\sum_{i=1}^{3}(\ddot{x}_i), \tag{2}$$

where $x_i \in \mathbb{R}[m]$ denotes the displacement of cart i, $\omega = \sqrt{\kappa/m}$ [rad/s] and $\zeta = \frac{\beta}{2m\omega}$ [-] are the angular eigenfrequency and the dimensionless damping coefficient of the carts, respectively. The parameters κ [N/m] and β [Ns/m] denote the (linear) stiffness and damping characteristics present in the system and m [kg] is the mass of the oscillator. On the other hand, the displacement of the coupling bar is described by $z \in \mathbb{R}$ [m]. Likewise, $\omega_z = \sqrt{\kappa_z/m_z}$ and $\zeta_z = \frac{\beta_z}{2m_z\omega_z}$ correspond to the angular eigenfrequency and the dimensionless damping coefficient of the coupling structure, respectively, where κ_z and β_z are the stiffness and damping coefficients of the coupling structure. The dimensionless small parameter $0 < \mu = \frac{m}{m_z} << 1$ denotes the *coupling strength*.

The actuation term U_i is designed by using a van der Pol term, i.e.

$$U_i = \nu(a_i x_i^2 - 1)\dot{x}_i, \qquad i = 1, 2, 3, \tag{3}$$

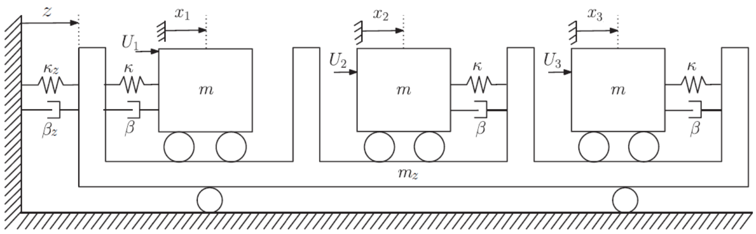


Figure 1: Generalized Huygens' setup: three actuated mass-spring damper oscillators mounted on a movable support.

where $\nu \in \mathbb{R}_+$ [1/s] determines the amount of nonlinearity and the strength of damping and $a_i \in \mathbb{R}_+$ [1/m^2] is a parameter which defines the switching between negative and positive damping for oscillator i. System (1)-(2) is analyzed by using the Poincaré method described in [4]. The analysis (not presented here due to space limitations) reveal that the coupled system (1)-(2) has the following (asymptotic) mixed synchronous solutions:

$$\lim_{t\to\infty} x_1(t) = -4r\sin\omega t, \quad \lim_{t\to\infty} x_2(t) = x_3(t) = 2r\sin\omega t, \quad \lim_{\to\infty} z = \dot{z} = 0, \tag{4}$$

with

$$r = \sqrt{\frac{\alpha - d}{\alpha a_3}}, \tag{5}$$

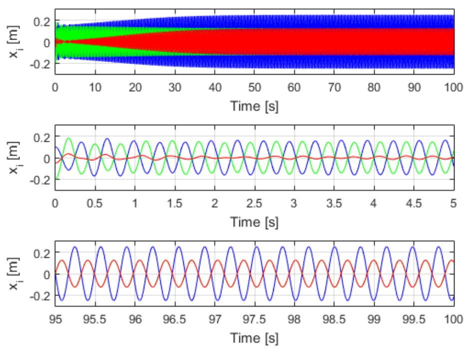
where $\alpha = \frac{\nu}{\mu\omega}$, $d = \frac{2\zeta}{\mu}$, and a_3 as given in (3).
The Poincaré method also reveals that the mixed synchronous solutions (4) exist and are asymptotically stable if and only if the following conditions are satisfied

$$\alpha > d, \qquad \text{and} \qquad a_1 = \frac{a_2}{4}, \qquad a_2 = a_3. \tag{6}$$

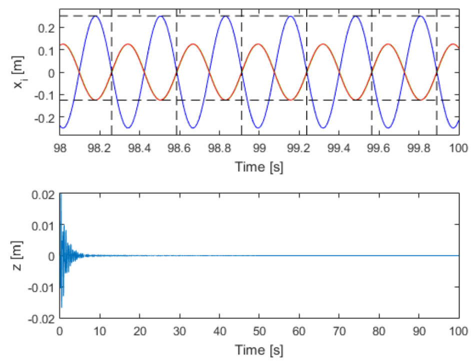
The mixed synchronous solutions (4) are very intuitive: in order to have mixed synchronization two oscillators must have the half of the amplitude of the other oscillator such that the forces exerted in the coupling bar cancel each other. Consequently, in the limit, the influence of the coupling system vanishes. On the other hand, note that conditions (6) have a physical interpretation: the first condition indicates that the amount of energy supplied to each oscillator should be larger than the amount of damping present in the oscillator. The second and third condition indicate that the oscilator 1 should be driven with a force which is twice the force exerted on the oscillators 2 and 3. Finally, note that the oscillation frequency of the mixed synchronous solutions (4) is exactly the same frequency corresponding to an uncoupled oscillator.

Numerical results

System (1)-(3) is numerically integrated by using the following parameter values: $\omega = 19.2634$ [rad/s], $\zeta = 0.0021$ [-], $\nu = 0.1926$ [kgm^2/rad^{3}s], $a_1 = 37.108$ [-], $a_2 = a_3 = 148.432$ [-] $\mu = 0.01$ [-], $\omega_z = 19.7157$ [rad/s], and $\zeta_z = 0.0828$ [-]. Note that the parameter values of the van der Pol term (3) satisfies conditions (6). The obtained results are shown in Figure 2a, which shows the time series of the simulation. The top window shows the complete simulation, while the middle and bottom window show the initial behavior and the steady synchronized behavior, respectively. Figure 2b shows the last two seconds of the simulation in the top window, whereas the bottom window shows the displacement of the coupling bar. The dotted horizontal lines in the top figure depict the amplitudes predicted from equations (4)-(5). On the other hand, the vertical dotted lines indicate the predicted period of the synchronous solution. Clearly, the numerical and analytical results coincide.



(a) Simulation results.



(b) Simulation results.

Conclusions and discussion

It has been shown that a triplet of mechanical oscillators interacting through Huygens' coupling can exhibit natural mixed synchronization, such that two oscillators synchronize in phase, whereas the other oscillator synchronize in anti/phase and with an amplitude which is twice the amplitude of the other two oscillators. The obtained results may be used in for example, the synthesis of controllers for the reduction or even elimination of vibrations in coupled (mechanical) systems.

References

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