Nonlinear Normal Modes of Coupled van der Pol Oscillators Exhibiting Synchronisation

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Summary. Nonlinear Normal Modes (NNMs), defined as two dimensional invariant manifolds in state space, have emerged as powerful analytical tools for the study of nonlinear systems. This work presents a novel approach to the study of synchronisation dynamics in mutually coupled van der Pol oscillators using the concept of NNMs. Shaw and Pierre's method is used to arrive at the NNMs as a two dimensional manifold parameterised by the displacement and velocity of the first oscillator. It is shown that because of the synchronising property of the system the invariant manifold reduces to a one-dimensional closed curve which remains a subset of the two-dimensional manifold calculated by NNM computation. This is shown to be true for both the oscillating modes of the system which corresponds to in-phase and out of phase synchronisation. It is also shown that the one-dimensional invariant manifold coincides with the synchronised limit cycle of the system for both modes. The NNMs are further used to decouple the governing equations. The decoupled equations which capture the modal dynamics retain the form of single van der Pol equation after removal of insignificant terms. This suggests a novel approach to the study of in-phase and out of phase synchronisation using these equations.

Introduction

Generalisation of the concept of normal modes to nonlinear systems, termed as Nonlinear Normal Modes (NNMs), have proved to be of tremendous practical interest. They have been used to investigate properties of nonlinear systems not obtained by other typical computational or analytical methods. Rosenberg [1] defined NNMs to be 'vibration in unison' where all material points of the system reach their extreme points at the same instant. Alternatively, NNM was defined as a two dimensional invariant manifold in state space by Shaw and Pierre [2] which can be parameterized by a single pair of state variables. Based on these two definitions, different analytical and computational strategies have been proposed to arrive at NNMs [3]. NNMs of coupled self-excited oscillators exhibiting synchronising behaviour has not received much attention in literature. Warminski [4] has studied NNMs of coupled Rayleigh oscillators using invariant manifold approach but has not considered the synchronising property of this system or its effects on NNMs. Synchronisation, defined as 'the adjustment of rhythms due to weak interaction', is a very well understood phenomenon [5] but it is surprising that its relationship to NNMs has not received much attention, given the close resemblance in definitions. This work tries to arrive at the NNMs of coupled van der Pol oscillators which exhibit mutual synchronisation and attempts to investigate the relationship between synchronisation and the concept of NNMs.

NNMs of coupled van der Pol oscillators

Consider two reactively coupled non-identical van der Pol oscillators. The governing equations of the system are

$$\begin{aligned}\ddot{x}_1 - (\mu_1 - {x_1}^2)\dot{x}_1 + {\omega_1}^2 x_1 + k(x_1 - x_2) = 0\\ \ddot{x}_2 - (\mu_2 - {x_2}^2)\dot{x}_2 + {\omega_2}^2 x_1 + k(x_2 - x_1) = 0\end{aligned} \tag{1}$$

Here, $x_{1,2}$ are displacements of the oscillators, k is the reactive coupling strength and $\omega_{1,2}$ are the limit cycle frequencies of the individual oscillators. We take the following numerical values for which the oscillators have been shown to be synchronised [5]: $\mu_{1,2} = 0.1$, $\omega_1 = 1$, $\omega_2 = 0.98$, and $k = 0.2$.The master coordinates are chosen as $x_1 = u$ and $x_2 = v$. Assume the functional form

$$x_2(u,v) = a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2 + a_6 u^3 + a_7 u^2 v + a_8 uv^2 + a_9 v^3 \tag{2}$$

$$y_2(u,v) = b_1 u + b_2 v + b_3 u^2 + b_4 uv + b_5 v^2 + b_6 u^3 + b_7 u^2 v + b_8 uv^2 + b_9 v^3 \tag{3}$$

Following Shaw and Pierre's technique and grouping the coefficients for each term results in 18 nonlinear algebraic equations in 18 variables. For the system parameters given above, this results in two sets of real solutions which correspond to different normal modes. Now, the parametric equation for the two dimensional manifolds can be written down explicitly. The first NNM, corresponding to in-phase oscillation, is given by the equations

$$x_2(u,v) = 1.1039u - 0.07u^3 - 0.079u^2 v - 0.0572uv^2 - 0.1213v^3 \tag{4}$$

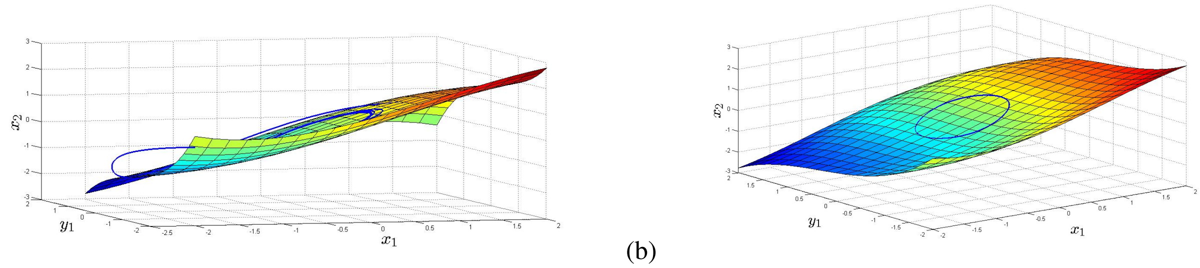
$$y_2(u,v) = 1.104v + 0.0777u^3 - 0.1072u^2 v + 0.1863uv^2 - 0.0935v^3 \tag{5}$$

The second NNM corresponding to the out of phase mode is specified by the equations

$$x_2(u,v) = -0.9059u - 0.047u^3 + 0.0962u^2 v - 0.039uv^2 + 0.0567v^3 \tag{6}$$

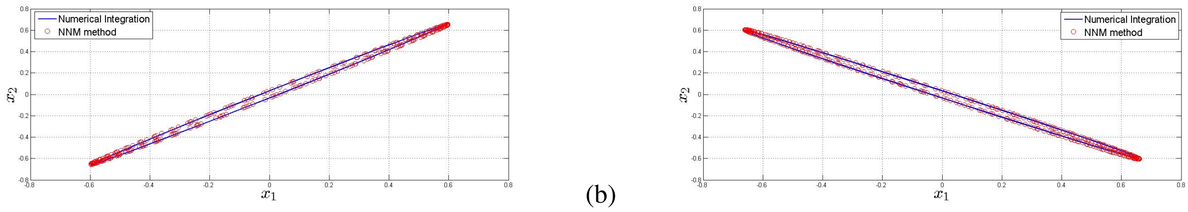
$$y_2(u,v) = -0.906v - 0.133u^3 - 0.237u^2 v - 0.0514uv^2 - 0.0219v^3 \tag{7}$$

Figure 1(a) shows the manifold corresponding to the in-phase mode using equation (4). The trajectory which is shown has its initial point on the manifold. It is clear that the manifold is not invariant under the flow. Figure 1(b) shows the steady state portion of the trajectory which is periodic. This orbit, which is the one dimensional invariant manifold, is seen to be a subset of the NNM manifold. The one dimensional subset of the NNM is in close agreement with the computationally calculated synchronised limit cycle as shown in figure 2.



(a) (b)

Figure 1: (a) The two dimensional manifold and a trajectory of the system starting from the manifold. (b) The periodic orbit as a subset of the manifold.



(a) (b)

Figure 2: Comparison between (a) in-phase and (b) out of phase modes using numerical intergation and NNM method.

Decoupling the governing equation

Inspite of not being invariant under the flow, the NNMs can still be used to decouple the governing equations (1) to obtain the equations pertaining to modal dynamics. In the present case the modal dynamics correspond to in-phase and out of phase synchronised oscillations. For deriving the equation corresponding to in-phase mode, equation (4) is plugged into the first equation in (1). For out of phase modal equation, equation (6) is used in the first equation in (1). This results in the modal equations which, after neglecting insignificant terms, gives

$$\begin{aligned} \ddot{u}_1 - (0.1 - {u_1}^2)\dot{u}_1 + 0.9866u_1 = 0 \\ \ddot{u}_2 - (0.1 - {u_2}^2)\dot{u}_2 + 1.1906u_2 = 0 \end{aligned} \tag{8}$$

Here, u_i's are the modal coordinates of the first mass, the in-phase mode being ${x_1}^{(1)} = u_1$ and the out of phase ${x_1}^{(2)} = u_2$. Comparing equation (8) with equation (1), it is clear that the effect of the coupling term, which is to synchronise the system, has been incorporated in the frequency term in the decoupled equations. For the in-phase mode, the frequency term has been modified to 0.9866 from 1 and for the out of phase case to 1.1906. These decoupled equations open up a novel way of studying synchronisation by solving these equations analytically.

Conclusions

This work attempts to arrive at the NNMs of coupled van der Pol oscillators. Synchronisation between the oscillators reduce the invariant manifold into a one dimensional closed curve which is identical to the synchronised limit cycle. The NNMs were used to decouple the equations. The decoupled equations retain the van der Pol form and can be solved analytically to analyse the system dynamics after synchronisation.

References

[1] Rosenberg R. M. (1962) The Normal Modes on Nonlinear n-degree-of-freedom Systems. *J. Applied Mechanics* **29**:7-14.

[2] Shaw S. W. and Pierre C. (1993) Normal Modes of Nonlinear Vibratory Systems.*J. Sound and Vibration* **164**:85-124.

[3] Kerschen G. et. al. (2009) Nonlinear Normal Modes, Part 1: A Useful Framework for Structural Dynamicist.*Mechanical Systems and Signal Processing* **23**:170-194.

[4] Warminski J. (2008) Nonlinear Normal Modes of Coupled Self-excited Oscillators in Regular and Chaotic Regimes.*J. Theoretical and Applied Mechanics* **46**:170-194.

[5] Balanov A. et. al. (2001) Synchronization: From Simple to Complex. Springer Verlag, Berlin..