Lyapunov stability of a planar rigid body with two frictional point contacts

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Summary. Lyapunov stability is highly desired in many mechanical engineering systems with unilateral contacts. Nevertheless, the Lyapunov stability analysis of equilibria is very difficult, because even small perturbations may result in hybrid dynamics with impacts and non-smooth transitions. Our work concerns with the analysis of a planar rigid body with two frictional unilateral point contacts under inelastic impacts. The dynamics of the system is examined via a low-dimensional projection of a Poincaré map. Our approach enables the determination of Lyapunov stability or instability for almost any equilibrium state. The results are illustrated by simulation examples and by stability and instability regions in two-dimensional parameter planes.

Background

Lyapunov stability of an equilibrium state is a fundamental concept in dynamical systems theory. For mechanical systems, it means that the dynamic response stays bounded in a small neighborhood of a static equilibrium configuration under small perturbations in the system's state, i.e. positions and velocities. This type of stability is highly desired in robotic applications such as grasping, quasistatic manipulation and legged locomotion, which commonly involve intermittent contacts. In (multi-) rigid body systems under unilateral frictional contacts, analysis of the dynamic response in the vicinity of an equilibrium state is a challenging task since it requires consideration of various transitions between different contact states, including separation, impacts and stick-slip transitions. In addition, it may suffer from difficulties such as solution indeterminacy or inconsistency due to Painlevé paradoxes [**?**, **?**].
Several earlier works have used the technique of Lyapunov functions, either energy-based or via sum-of-squares optimization, in order to prove stability and find bounds on region of attraction [**?**, **?**]. Nevertheless, a major drawback of this technique is that it can only prove stability but does not enable determination of instability. Other works utilized the Poincaré map approach, typically by obtaining the discrete-time dynamics of states at impact times. These works were limited to systems with a single contact and ignored slippage and friction constraints, thus considering trivial hybrid transitions.

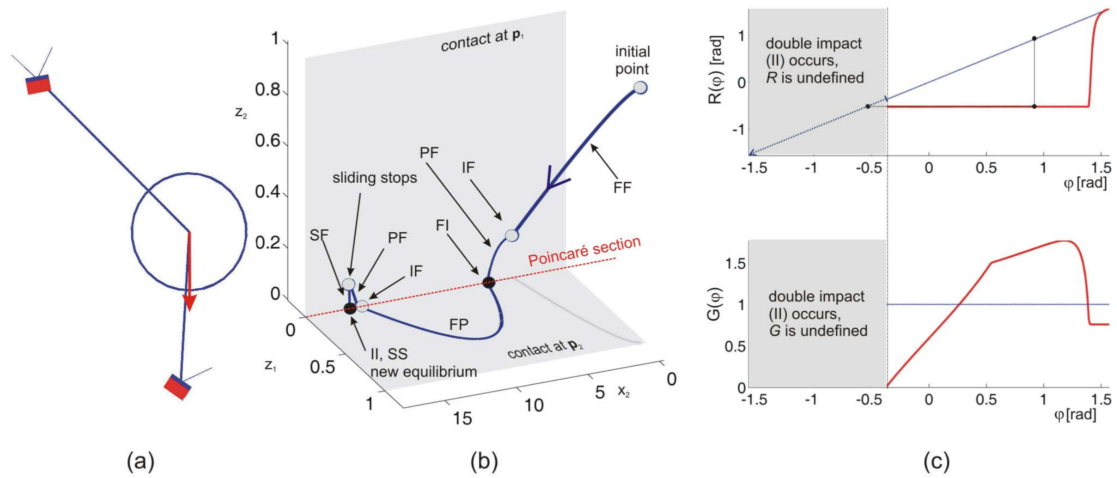

Figure 1: (a) Two-contact equilibrium configuration. The circle represents the radius of gyration and the position of the center of mass. The arrow is an external force. The friction cones at the contact points are shown graphically. (b) A solution trajectory with two-letter labels representing contact modes of the two points. x_2, z_1 and z_2 is a set of generalized coordinates (details omitted). The meanings of the letters are positive (P) or negative (N) slip, stick (S), free motion (F), or impact (I). IF and FI means impacts at one point while the other switches to F mode. II is a simultaneous impact at the two points. (c) Plots of the reduced Poincaré map $R(\varphi)$ (top) and the growth map $G(\varphi)$ (bottom). φ is the angle of the pre-impact normal velocity at point 2 with respect to the contact normal and $R(\varphi)$ gives the value of this angle after a full cycle. The function G encodes the factor of growth of the normal velocity of point 2 in one full Poincaré cycle. Our detailed analysis reveals that both functions may be undefined at some points, which is related to simultaneous impacts (II) events.

Problem statement, and solution methods

In this work [**?**], we revisit the simplest possible example of a non-trivial mechanical contact system earlier studied by [**?**, **?**]: a planar rigid body with two unilateral frictional contacts under inelastic impacts (Fig. 1(a)). The motion of the

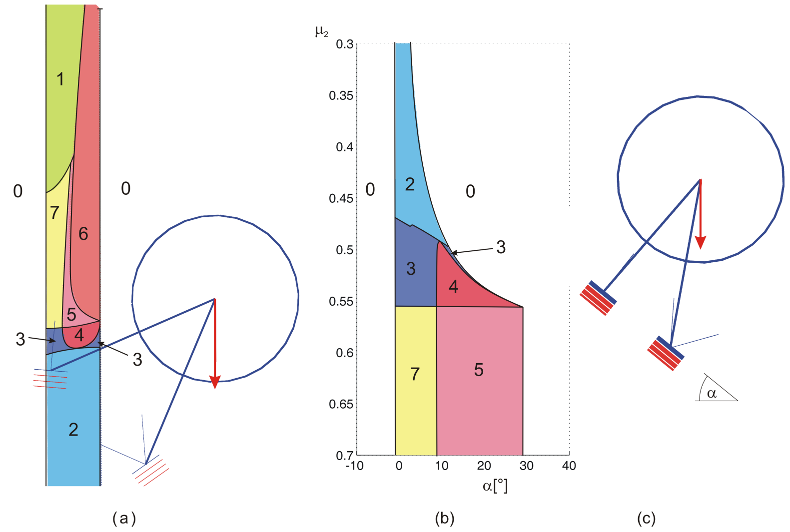

Figure 2: (a) Example 1 – regions of center-of-mass position with different stability properties plotted over a two-contact configuration. (b,c) Example 2 – regions with different stability properties in (α, μ_2) plane and a nominal two-contact configuration. The meanings of the enumerated regions are: not an equilibrium (0); excluded from the analysis due to Painlevé's paradox (1) or due to a technical constraint, which will be removed by future work (7); Lyapunov stable (2,3); unstable due to reverse chatter (4,5) or due to ambiguity (6).

system is approximated by its *zero-order dynamics* where the solution under each possible contact mode involves constant accelerations and contact forces (see Fig. 1(b) for a simulation examples). A simple impact model is used, which takes the form of a piecewise linear mapping of generalized velocities. We define a Poincaré map for states of impact at one contact point and sustained contact at the other (Fig. 1(b)). Exploiting invariance relations, this three-dimensional Poincaré map is reduced into a scalar map R and a scalar growth function G which together encapsulate the entire dynamics of the system under any local perturbation (Fig. 1(c)).

Conditions of stability and instability

Two possible mechanisms of instability are identified: one due to *ambiguous equilibrium* (i.e. the coexistence of the equilibrium with a non-static solution) and the other due to a *reverse chattering* (an infinite, diverging sequence of impacts resembling the motion of a bouncing ball backwards in time). At the same time, we also find evidence of finite-time convergence to equilibrium in other situations via an infinite decaying sequence of impacts (like a bouncing ball forward in time) or via a finite sequence involving a simultaneous impact at the two points. Based on these observations, we present a semi-analytic method for determination of Lyapunov stability or instability for almost any possible two-contact equilibrium state satisfying some mild conditions, by analyzing the interval graph structure of the reduced Poincaré map.

Conclusions

Our results show that the Lyapunov stability of the object depends on model parameters in a highly nontrivial and often counterintuitive way. In Fig. 2, we show two examples of stability regions in two-dimensional sections of the space of model parameters. In panel (a) the position of the center of mass is varied, whereas panel (b) shows, for an object resting on a slope (see panel(c)), the effects of slope angle and the friction coefficent at one of the two contact points. Counterintuitive features of these plots include for example that increasing the angle α may either stabilize or destabilize the object.

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