Numerical approximation of invariant manifolds for dynamical systems with simultaneous self- and forced excitation

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Summary. Since a broad range of engineering problems is simultaneously excited by different physical origins, quasiperiodic oscillations can often be observed. This work deals with a method to approximate invariant manifolds of periodic and quasiperiodic motions by means of a numerical approach. In order to give an alternative to the conventional approach a partitioning of the phase variables is proposed, which separates torus coordinates from dependent phase angles. To demonstrate the use of this approach the method is applied to a textbook example, the forced VAN DER POL equation, and a practical example, the unbalanced LAVAL-rotor (JEFFCOTT-rotor). Moreover, a criterion for the attractiveness of invariant manifolds is discussed.

Introduction

In contrast to periodic motions, so far the analysis of quasiperiodic oscillations has not yet become a standard in vibrational analysis of engineering problems. However, the broad range of self-excited systems subjected to external forcing systematically produces constellations, where oscillations of different physical origin may interfere: in this context, rotor dynamical problems involving self-excitation and forcing due to unbalance are mentioned as an example of great practical relevance. In general, the presence of different (incommensurable) physical excitation frequencies may lead to quasiperiodic oscillations unless synchronisation occurs.

This paper aims on a systematic approach to determine stationary quasiperiodic motions by approximating the corresponding invariant manifold. Eventually, this approach may be the basis for a continuation of stationary solutions as well as investigations of the stability of such stationary motions. In order to describe a quasiperiodic oscillation an invariant manifold is calculated, whose parametrisation is defined by torus coordinates. Therefore, it is required that the equations of motion are transformed to amplitude and phase variables. The conventional approach [4, 6, 7] provides a realizable method to calculate invariant manifolds by choosing a canonical form and assuming parallel flow. A disadvantage which arises from these assumptions is that internal frequencies of a system have to be known. By adding phase conditions, which characterize the internal frequencies of free phases, a unique solution can be calculated [4]. These conditions use the orthogonality in suitable SOBOLEV spaces of a solution and the derivative of a previous solution of a continuation step.

In contrast to the conventional approach, a method is proposed, which aims on solving the primary invariance equation of dynamical systems without the assumption of a parallel flow and the necessity of knowing a previous solution.

Method

Assume an autonomous dynamical system given in amplitude and phase variables. In most cases the parametrisation of the invariant torus will be much smaller than the number of degrees of freedom of the system. To guarantee a solution for dynamical systems with arbitrary internal frequencies and thus make the proposed method applicable to large problems, the phase angles are partitioned into torus coordinates and dependent phase angles. Hence the problem has the structure

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}), \qquad \mathbf{z} = \begin{bmatrix} \mathbf{A} \\ \boldsymbol{\theta} \\ \boldsymbol{\varphi} \end{bmatrix}, \qquad \mathbf{f}(\mathbf{z}) = \begin{bmatrix} \mathbf{h}(\mathbf{z}) \\ \boldsymbol{\Omega}(\mathbf{z}) \\ \boldsymbol{\Psi}(\mathbf{z}) \end{bmatrix}, \tag{1}$$

where $\boldsymbol{\theta}$ is the vector of torus coordinates, $\mathbf{A} = \mathbf{A}(\boldsymbol{\theta})$ is the vector of amplitudes and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{\theta})$ is the vector of dependent phase angles. The dependent and independent phase angles (torus coordinates) are selected according to the origin of excitation, whereby the identification is not unique. For example, the phase angles which describe self-excited motions can either be independent or dependent phase angles, where the deciding factors are the occurrence of synchronisation or mechanisms of self-excitation which affect multiple phase angles at once. If one takes a system which is only excited by external forces, each phase angle describes an independent phase angle as long as it's frequency is incommensurable to the others. Because of this partitioning of the phase angles the invariant manifold is defined by

$$\mathcal{M} = \{(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \mid \mathbf{A} = \mathbf{A}(\boldsymbol{\theta}),\ \boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \mathbb{T}^p,\ \mathbf{A} \in \mathbb{R}^q,\ \boldsymbol{\varphi} \in \mathbb{R}^r\} \ \subset\ \mathbb{T}^p \times \mathbb{R}^q \times \mathbb{R}^r, \tag{2}$$

where the p-dimensional standard torus $\mathbb{T}^p$ is the set which is parametrised over $(0, 2\pi]^p$. Furthermore, the invariant manifold is only invariant, if the vector of time derivations $[\mathbf{h}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}), \boldsymbol{\Omega}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}), \boldsymbol{\Psi}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi})]^{\mathrm{T}}$ lies in the tangent space $\mathcal{T}_{(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi})}\mathcal{M}$ for all times. Since the basis of the tangent space is defined by

$$\mathbf{j}_k = \left(\frac{\partial A_1}{\partial \theta_k}, ..., \frac{\partial A_q}{\partial \theta_k},\ \underbrace{0, ..., 1, ..., 0}_{k-\text{th unit vector } \mathbf{e}_k},\ \frac{\partial \varphi_1}{\partial \theta_k}, ..., \frac{\partial \varphi_r}{\partial \theta_k}, \right)^{\mathrm{T}} \in \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^r, \qquad k = 1, ..., p \tag{3}$$

the vector of the time derivations can be expressed as a linear combination of the basis vectors

$$\sum_{k=1}^{p} \alpha_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi})\mathbf{j}_k = \begin{bmatrix} \mathbf{h}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \\ \boldsymbol{\Omega}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \\ \boldsymbol{\Psi}(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \end{bmatrix} \tag{4}$$

considering coefficient functions $\alpha_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi})$. Because of the k-th unit vector in each basis vector, the rows $q + k,\ k = 1, ..., p$ of (4) lead to the identification of the coefficient functions $\alpha_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) = \Omega_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi})$. Substituting the coefficient functions, a system of partial differential equations (invariance equations) results whose solution in terms of amplitudes and dependent phase angles describes the invariant manifold. The resulting boundary value problem reads

$$\begin{bmatrix} \sum\limits_{k=1}^{p} \dfrac{\partial A_i}{\partial \theta_k} \Omega_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \\ \\ \sum\limits_{k=1}^{p} \dfrac{\partial \varphi_j}{\partial \theta_k} \Omega_k(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \end{bmatrix} = \begin{bmatrix} h_i(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \\ \\ \Psi_j(\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\varphi}) \end{bmatrix}, \qquad \boldsymbol{\theta} \in \mathbb{T}^p \quad \text{with} \quad i = 1, ...q, \quad j = 1, ...r, \tag{5}$$

and if the area of definition is the interval $[0, 2\pi]^p$, the periodic boundary conditions

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta} + 2\pi\mathbf{e}_k), \quad \boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\varphi}(\boldsymbol{\theta} + 2\pi\mathbf{e}_k), \qquad k = 1, ..., p \tag{6}$$

apply. The advantages of distinguishing between dependent and independent phase angles are on the one hand that a periodic motion can be described over a manifold and on the other hand a quasiperiodic oscillation of a system within dependent phase angles can be solved. Furthermore, the stability of stationary solution is irrelevant to the numerical algorithm, since the invariance equations are solved. The system of partial differential equations (5) is in the context of this work discretized with a standard finite difference method, where the order of derivatives can be changed depending on the needed accuracy.

Results

In order to demonstrate the proposed extensions, the method is firstly applied to the forced VAN DER POL equation. Since this equation can oscillate with a stationary periodic and quasiperiodic motion and possesses only two phase variables, the application of the proposed method is easily comprehensible. Moreover, a criteria is proposed which can identify the attractiveness of stationary motions, with focus on the quasiperiodic one. Secondly, the method is applied to a complex example, which contains dependent phase angles regardless of a stationary periodic or quasiperiodic oscillation: An unbalanced LAVAL-rotor (JEFFCOTT-rotor) with linear internal damping and linear and nonlinear outer damping.

Forced VAN DER POL equation

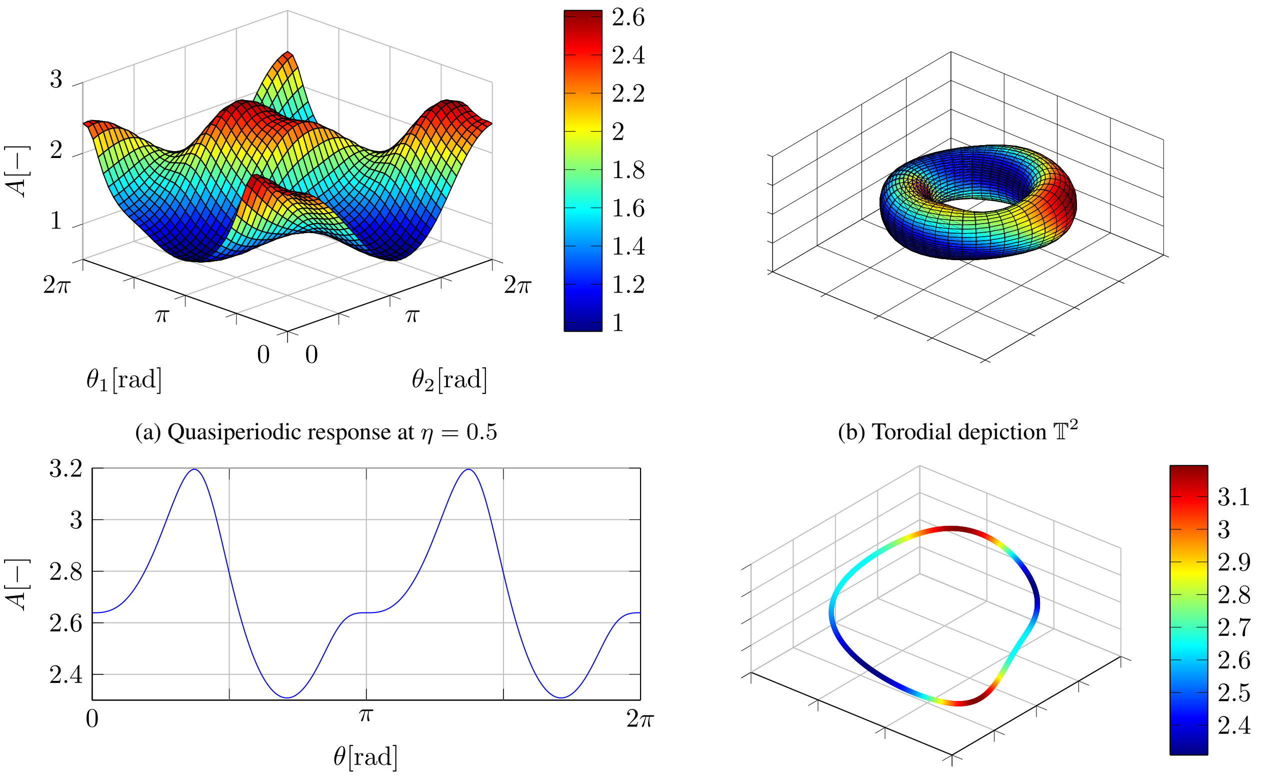
As a first example the invariant manifold of a stationary solution of the forced VAN DER POL equation, given by the nondimensional equation

$$q'' + \varepsilon\left(q^2 - 1\right)q' + q = \Gamma\cos(\eta\tau), \tag{7}$$

is calculated. By transforming the equation of motion into the state-space $[q, q']^{\mathrm{T}} = [z_1, z_2]^{\mathrm{T}}$ with a subsequent transformation in amplitude and phase $[z_1, z_2]^{\mathrm{T}} = [A\cos(\gamma_1), A\sin(\gamma_1)]^{\mathrm{T}}$, the equations read

$$\begin{bmatrix} A' \\ \gamma_1' \\ \gamma_2' \end{bmatrix} = \begin{bmatrix} -\varepsilon\left[A^2\cos^2(\gamma_1) - 1\right]A\sin^2(\gamma_1) + \Gamma\cos(\gamma_2)\sin(\gamma_1) \\ -1 - \varepsilon\left[A^2\cos^2(\gamma_1) - 1\right]\sin(\gamma_1)\cos(\gamma_1) + \dfrac{\Gamma}{A}\cos(\gamma_2)\cos(\gamma_1) \\ \eta \end{bmatrix}, \tag{8}$$

where γ_2 has been introduced to cast the problem in autonomous form. Performing a classic linear analysis of system (8) leads to the result, that an area of periodic orbits exists around $\eta \approx 1$, where the self-excitation frequency synchronise to the forced excitation frequency [2, 3]. In immediate vicinity of this area a multifrequent oscillation exists, since the detuning of the excitation frequencies is too large. Regarding the proposed method a system with a periodic $\left([A, \gamma_1, \gamma_2]^{\mathrm{T}} = [A, \varphi, \theta]^{\mathrm{T}}\right)$ response exists in the area of synchronisation, in which the choice of dependent and independent phase angle is arbitrary since it occurs in the 1:1 resonance. Besides the area of synchronisation a quasiperiodic $\left([A, \gamma_1, \gamma_2]^{\mathrm{T}} = [A, \theta_1, \theta_2]^{\mathrm{T}}\right)$ response occurs, where both phase angles are torus coordinates. Taking the characteristics of the periodic and quasiperiodic motion concerning dependent and independent phase angles and formulating the system of partial differential equations under the consideration of (5) lead to the manifolds depicted in figure 1.



(a) Quasiperiodic response at $\eta = 0.5$

(b) Torodial depiction $\mathbb{T}^2$

(c) Periodic response at $\eta = 1$

(d) Torodial depiction $\mathbb{T}^1$

Figure 1: Calculated manifolds for a periodic and quasiperiodic response of the VAN DER POL equation. ($\varepsilon = 0.3$ and $\Gamma = 0.6$)

Identifying the stability of such a quasiperiodic oscillation is possible from a theoretical point of view [5], but the application to an arbitrary differential equation is generally not possible. A transformation of variables can lead to stability criteria, in the case of two strong coupled VAN DER POL equations see [8], but the existence or the identification of such a transformation is not guaranteed.

Because of the latter an investigation of the attractiveness of an invariant stationary solution is proposed, in which the invariance equations (5) is solved numerically as an initial value problem based on a calculated manifold. The idea is to use the property of the state-space for a sufficiently smooth quasiperiodic solution $\mathbf{z} \in \mathcal{C}^r(\mathbb{T}^p), r \geq 1$, that a crossing of orbits $\gamma = \{\mathbf{z} \mid \mathbf{z} = \mathbf{z}(\tau, \mathbf{z}_0), \tau \in \mathbb{R}\}$ is impossible for autonomous systems. Initially one defines the boundaries of a calculated solution and an arbitrary finite perturbation

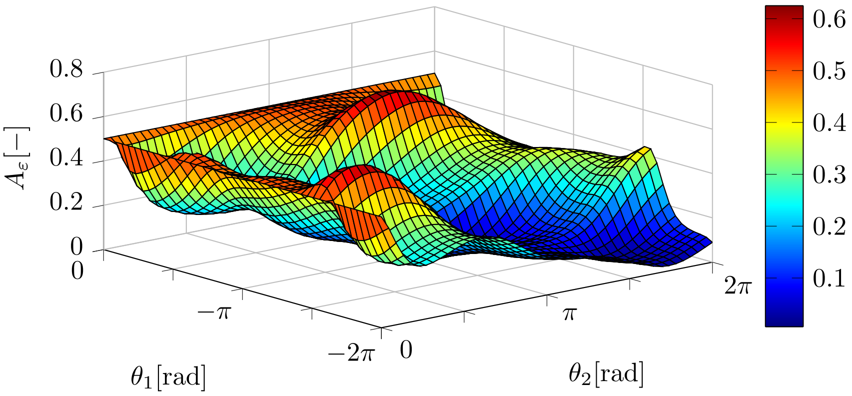
$$\begin{aligned} &\partial A_{\mathcal{M}}^{(i)}(\theta_1, ..., \theta_i, ..., \theta_p) \\ &\partial A_{\varepsilon}^{(i)}(\theta_1, ..., \theta_i, ..., \theta_p) \end{aligned}, \quad i = 1, ..., p\,, \quad \theta_i = \text{const.}, \tag{9}$$

whereby $A_{\mathcal{M}}$ is the solution of the manifold and A_ε the perturbation. By adding the perturbation to the manifold on the boundaries and solving the equations over the interval $(0, 2\pi]^p$, the sufficient criterion

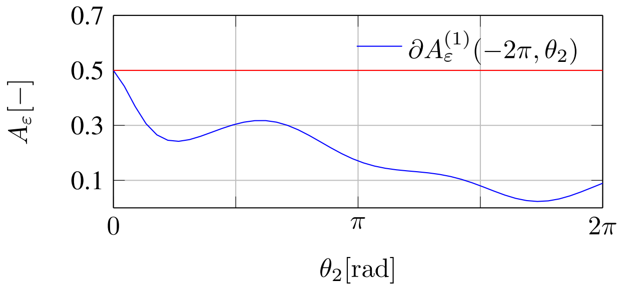
$$\left|\partial A_{\varepsilon}^{(i)}(\theta_1, ..., \theta_i, ..., \theta_p)\right| > \left|\partial A_{\varepsilon}^{(i)}(\theta_1, ..., \theta_i \pm 2\pi, ..., \theta_p)\right| \quad \text{with} \quad i = 1, ...p\,, \tag{10}$$

for the attractiveness of the manifold results. The choice of the algebraic sign depends on the physical direction of the torus coordinates concerning the time variable. If a finite perturbation exists which fulfils equation (10), the maximum (minimum) amplitude of the calculated initial value problem is the maximum (minimum) amplitude which the system can achieve for $\tau \to \infty$, since the calculated solution represents all possible trajectories. Due to the 2π-periodicity of the transformed state-space a numerical investigation of a finite area is sufficient to make a statement about the attractiveness of the invariant manifold.

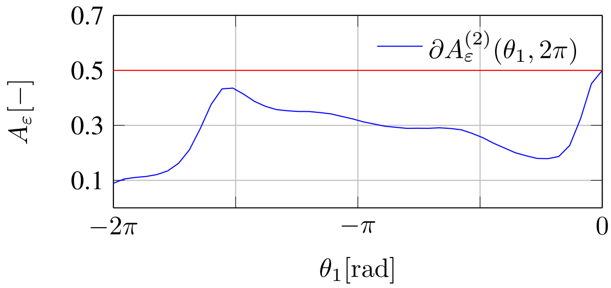
To apply the proposed criterion for the attractiveness on the calculated quasiperiodic manifold depicted in figure 1a, the solution of the initial value problem is required.



(a) Depiction of A_ε in the parametrised space



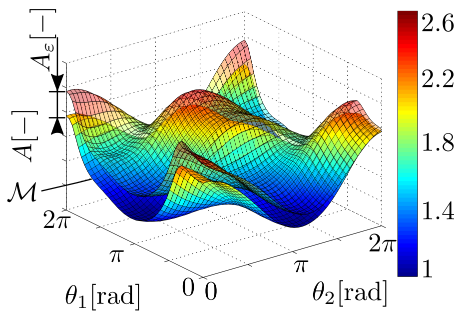
(b) θ_1-A_ε-plane



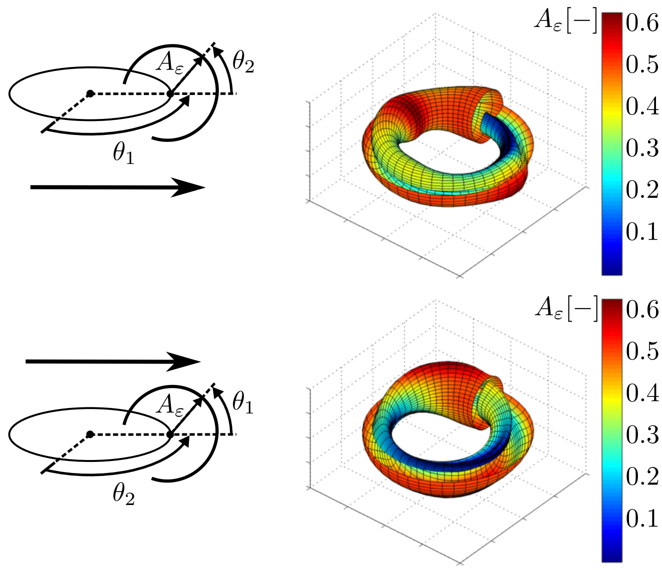
(c) θ_2-A_ε-plane

Figure 2: Solution of the initial value problem based on the invariant manifold depicted in figure 1a with constant perturbations $\partial A_\varepsilon^{(1)}(0,\theta_2) = \partial A_\varepsilon^{(2)}(\theta_1, 0) = 0.5$ on the boundaries $\partial A_{\mathcal{M}}^{(1)}(0,\theta_2)$ and $\partial A_{\mathcal{M}}^{(2)}(\theta_1, 0)$.

In figure 2a is the evolution of the constant perturbation $\partial A_\varepsilon^{(1)}(0,\theta_2) = \partial A_\varepsilon^{(2)}(\theta_1, 0) = 0.5$ over the parametrised space depicted. Considering figure 2b and figure 2c, one can easily see that the quasiperiodic solution fulfils condition (10). Since the dynamics of the system is periodic over the parametrisation, one can imagine a modified initial value problem starting with the boundaries depicted in figure 2b and figure 2c lying under the calculated perturbation illustrated in figure 2a. The following result can only exist between the primarily calculated solution and $A_\varepsilon = 0$, which corresponds to a solution on the manifold. The increase of the solution over the primarily calculated solution or the decrease under $A_\varepsilon = 0$ implies an impossible crossing of trajectories in the state-space.



(a)



(b)

Figure 3: Results for the forced VAN DER POL equation: (a) Invariant manifold $\mathcal{M}$ and the solution of the initial value problem (transparent). (b) Toroidal representation of the amplitude difference (A_ε-torus) specified in (a)

A geometrical interpretation of the described approach by a toroidal representation is realizable for a $\mathbb{T}^2$ and depicted in figure 3. If the manifold is asymptotically stable, the perturbation has the characteristic $A_\varepsilon \to 0\ \forall(\theta_1, \theta_2)$ for $\tau \to \infty$, which is a flat plain in the state-space. By choosing a toroidal depiction a ring results, which implies that the perturbation has to decrease over the parametrisation. Furthermore, if the torus decreases over an interval $[0, 2\pi]$ an expansion through the enclosed solution corresponds to a crossing of trajectories. Considering figure 3 a decrease of the A_ε-torus over the parametrisation is observable, whereby with the given initial constant perturbation on the boundaries, the maximum amplitude stays below the maximum value $A_\varepsilon \approx 0.625$ for $\tau \to \infty$.

Unbalanced LAVAL-rotor

As a second example the rotor dynamics of an unbalanced LAVAL-rotor with linear internal, linear outer and nonlinear outer damping is analysed. The additional nonlinear terms represent a simple approximation of nonlinear damping forces, which appear in squeeze oil dampers. These systems are of interest in the context of the proposed method since they have the ability to oscillate with a stationary quasiperiodic motion [1, 9, 10]. The equations of motion read

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}y_S''\\ z_S''\end{bmatrix} + \begin{bmatrix}2(D_a + D_i) & 0\\ 0 & 2(D_a + D_i)\end{bmatrix}\begin{bmatrix}y_S'\\ z_S'\end{bmatrix} + \begin{bmatrix}D_{NL} & 0\\ 0 & D_{NL}\end{bmatrix}\begin{bmatrix}y_S'^3\\ z_S'^3\end{bmatrix} + \begin{bmatrix}1 & 2D_i\eta\\ -2D_i\eta & 1\end{bmatrix}\begin{bmatrix}y_S\\ z_S\end{bmatrix} = \begin{bmatrix}e\eta^2\cos(\eta\tau)\\ e\eta^2\sin(\eta\tau)\end{bmatrix}, \tag{11}$$

where y_S and z_S are the coordinates describing the displacement of the shaft, D_a is the coefficient of the linear outer damping, D_i is the coefficient of the linear internal damping, η describes the rotation frequency, e is the eccentricity of the center of mass and D_{NL} is the coefficient of a cubic nonlinear outer damping. In contrast to the linear example, these terms ensure a stationary solution in the area where self-excitation occurs. By continuing the NEIMARK-SACKER-bifurcation with MATCONT a dependence of the stability limit on the eccentricity of the center of mass is detected.

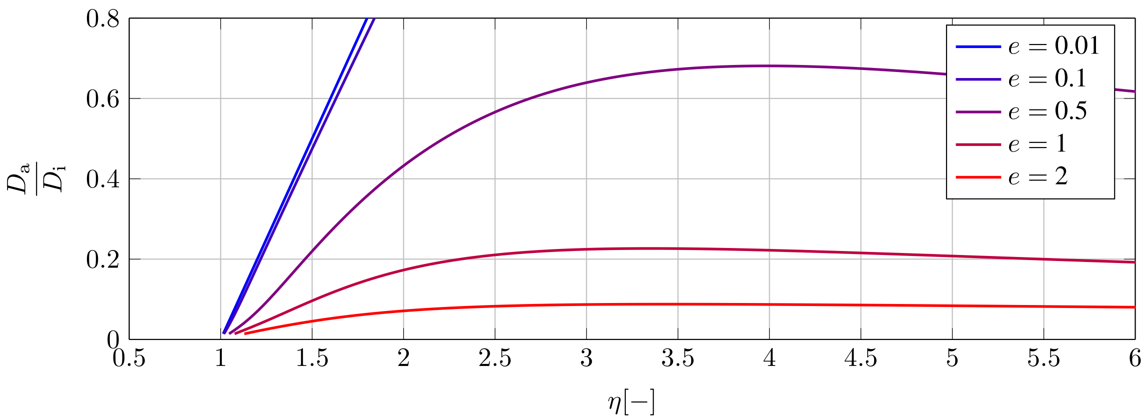


Figure 4: Stability chart of the model for the nonlinear LAVAL-rotor varying the eccentricity e ($D_{NL} = 0.07$).

On the stability limit a NEIMARK-SACKER-bifurcation occurs, the periodic oscillation loses its stability and a quasiperiodic solution arises. Through the linear analysis of the homogeneous model without nonlinear outer damping, it is well known at which frequencies self-excitation occurs. If the eccentricity is zero, the stability limit of the nonlinear model is equal to a linear model of the LAVAL-rotor. As long as the magnitude of the eccentricity is relatively small compared to the magnitude of the frequency, the forced excitation is weak and can only perturb the frequency of self-excitation close to the stability limit strong enough to synchronise. By increasing the eccentricity with the same ratio of linear outer and internal damping the stability limit is shifted to higher excitation frequencies, since the perturbation on the frequency of self-excitation rises and the area of a stable periodic solution increases. Identifying the sources of excitation one distinguishes a self-excitation due to circulatory terms, which depends linearly on the frequency ($2D_i\eta$), and a forced excitation due to the unbalanced center of mass, which depends quadratically on the frequency $(e\eta^2\cos(\eta\tau), e\eta^2\sin(\eta\tau))$. If the eccentricity reaches a certain value at a constant ratio of linear outer and internal damping the frequency of the self-excitation is always synchronised to the forced excitation frequency due to the different dependence of the strength of excitation on the frequency.

To analyse the model defined by equations (11) with the proposed method, the equations of motion are transformed into the state-space $[y_S, y_S', z_S, z_S']^T = [z_1, z_2, z_3, z_4]^T$. Choosing an amplitude and phase representation of the state-space variables $[z_1, z_2]^T = [A_1\cos(\gamma_1), A_1\sin(\gamma_1)]^T$ and $[z_3, z_4]^T = [A_2\cos(\gamma_2), A_2\sin(\gamma_2)]^T$ and introducing the excitation phase $\eta\tau = \gamma_3$, the system is rewritten in autonomous from:

$$\begin{bmatrix} A_1' \\ A_2' \\ \gamma_1' \\ \gamma_2' \\ \gamma_3' \end{bmatrix} = \begin{bmatrix} \sin(\gamma_1)\left[e\eta^2\cos(\gamma_3) - 2D_\mathrm{i}\eta A_2\cos(\gamma_2) - A_1\sin(\gamma_1)\left\{2(D_\mathrm{a}+D_\mathrm{i}) + D_\mathrm{NL}A_1^2\sin^2(\gamma_1)\right\}\right] \\ \sin(\gamma_2)\left[e\eta^2\sin(\gamma_3) + 2D_\mathrm{i}\eta A_1\cos(\gamma_1) - A_2\sin(\gamma_2)\left\{2(D_\mathrm{a}+D_\mathrm{i}) + D_\mathrm{NL}A_2^2\sin^2(\gamma_2)\right\}\right] \\ -1+\cos(\gamma_1)\left[\frac{e\eta^2}{A_1}\cos(\gamma_3) - \frac{2D_\mathrm{i}\eta A_2}{A_1}\cos(\gamma_2) - \sin(\gamma_1)\left\{2(D_\mathrm{a}+D_\mathrm{i}) + D_\mathrm{NL}A_1^2\sin^2(\gamma_1)\right\}\right] \\ -1+\cos(\gamma_2)\left[\frac{e\eta^2}{A_2}\sin(\gamma_3) + \frac{2D_\mathrm{i}\eta A_1}{A_2}\cos(\gamma_1) - \sin(\gamma_2)\left\{2(D_\mathrm{a}+D_\mathrm{i}) + D_\mathrm{NL}A_2^2\sin^2(\gamma_2)\right\}\right] \\ \eta \end{bmatrix}. \tag{12}$$

With regard to figure 4 there are two main types of behaviour in this system. On the left hand side of the stability limit exists a periodic orbit where the source of excitation is due to a force. Concerning the phase angles there is only one torus coordinate and two dependent phase angles since all phases move with the same frequency $[A_1, A_2, \gamma_1, \gamma_2, \gamma_3]^\mathrm{T} = [A_1, A_2, \varphi_1, \varphi_2, \theta]^\mathrm{T}$. On the right hand side exists a simultaneous self- and forced excitation which results in a quasiperiodic orbit. Since the second source of excitation belongs to the self-excitation the phase angles γ_1 and γ_2 both move with the same frequency, therefore only one of them is a torus coordinate and the other a dependent phase angle. The choice is because of the latter arbitrary, which leads to $[A_1, A_2, \gamma_1, \gamma_2, \gamma_3]^\mathrm{T} = [A_1, A_2, \theta_2, \varphi, \theta_1]^\mathrm{T}$. Applying the proposed method allows a continuation of the periodic and quasiperiodic solution which are depicted in figure 5 . For the one parameter continuation with respect to η, the remaining values are fixed $D_\mathrm{i} = 1, D_\mathrm{a} = 0.07, D_\mathrm{NL} = 0.07$ and $e = 2$.

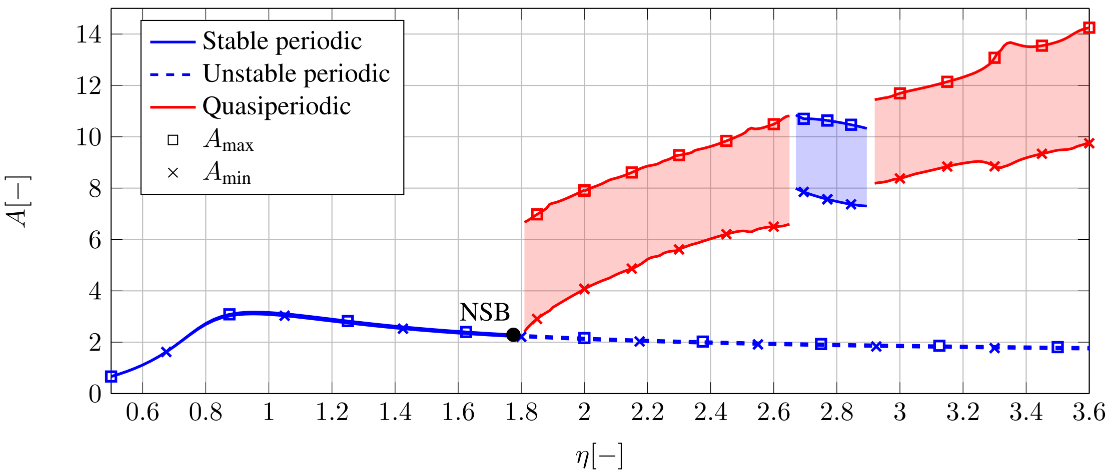


Figure 5: Continuation of periodic and quasiperiodic solutions $\left(A = \sqrt{y_\mathrm{S}^2 + z_\mathrm{S}^2}\right)$.

The continuation of the periodic solution is initialised by computing a solution at $\eta = 0.5$ and continued to $\eta = 3.6$. Since parametrisation is over $[0, 2\pi]$, the mesh is one dimensional and 100 nodes are chosen. The periodic solution is stable until a supercritical NEIMARK-SACKER-bifurcation ($\eta \approx 1.8$) occurs and the periodic solution loses its stability. To calculate the quasiperiodic solution an initial parameter value ($\eta = 2$), which is far enough away from the periodic solution, and a 50 x 50 mesh is chosen. Furthermore a step width of $\eta = 0.025$ is selected. By decreasing η the proposed method can continue the quasiperiodic solution close to the NEIMARK-SACKER-bifurcation, but because of the torus collapse not reach it. During the continuation there probably exist other resonances, but either their large periods make them numerically quasiperiodic or the synchronisation range is to small for the chosen step width. If one increases η the algorithm converges and continues the periodic solution up to $\eta \approx 2.7$, where no solution is found. By starting a continuation at $\eta = 3.6$ and decreasing η the algorithm converges until $\eta \approx 2.9$. A waterfall chart of the area of interest reveals a 1:3 synchronisation (figure 6).

Starting a continuation of a periodic orbit at $\eta \approx 2.77$ with subsequent decreasing (or increasing) of η until the algorithm does not converge, leads to the periodic solution depicted in figure 6. The proposed method is able to calculate the periodic solution, but since in a phase locked state a stable and unstable periodic orbit exists, where both are invariant, the choice of suitable initial conditions for a unique identification is still a task.

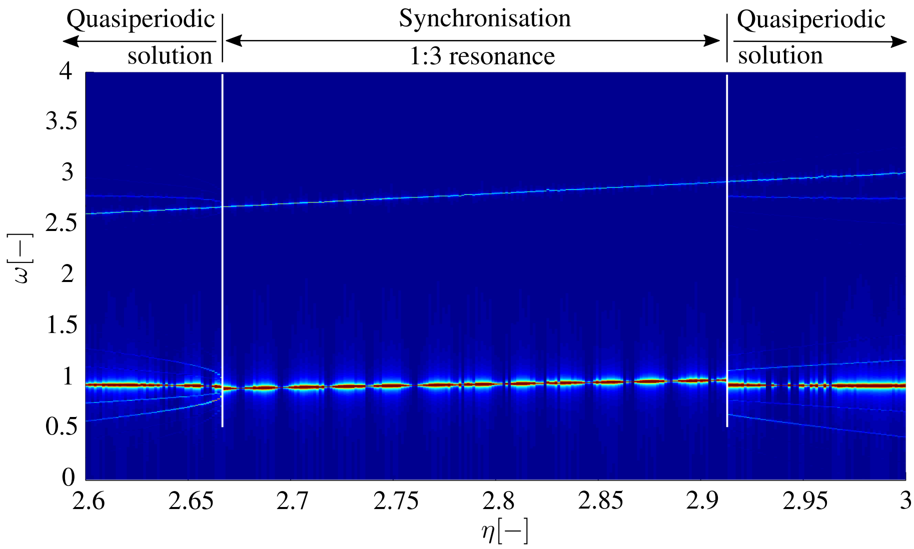


Figure 6: Waterfall chart showing the 1:3 synchronisation within a range of $\eta \approx 2.66$ to $\eta \approx 2.91$.

During a continuation of a quasiperiodic solution one can expect three possible cases. The first case is an occurrence of a bifurcation which leads to a torus collapse. The second case is a synchronisation of the excitation frequencies, which leads to simultaneous existing invariant solutions. Last of all a coincidental incommensurability of the excitation frequencies can occur, which does not cause a torus collapse, whereby it is numerical irrelevant.

Conclusion

In this paper a method to calculate invariant manifolds of stationary motions is proposed, which is based on solving the primary invariance equation deduced in [5]. With a partitioning of the phases in dependent and independent phase angles a unique solution can be calculated over the parametrised interval $[0, 2\pi]^p$ also for quite general systems, having much more degrees of freedom than torus coordinates. In contrast to the conventional approach one does not need phase conditions or a solution of a previous time step. However, the proposed method implies the task of knowing the sources of excitation. To demonstrate the usability, the proposed method is applied to the forced VAN DER POL equation. Since this equation can exhibit periodic and quasiperiodic oscillations only having two phase angles, the partitioning can easily be comprehended. Furthermore, a criterion to evaluate the attractiveness of stationary motions is proposed and applied in the context of this example. The practicability of the proposed method is shown by applying it to an unbalanced LAVAL-rotor. In this example an extended area of stability can be identified through the presence of eccentricity. By continuing the quasiperiodic motion a 1:3 synchronisation is found and the invariant periodic solution is calculated with the proposed method.

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