Chattering motion of rigid objects

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Summary. Ideally rigid objects establish sustained contact with one another via an infinite sequence of collisions accumulating in finite time (complete chatter). Alternatively, such systems may also exhibit a finite sequence of collisions followed by separation (incomplete chatter). Motivated by existing results concerning the 2D chattering of slender rods, we examine this phenomenon in the case of three-dimensional objects with multiple points hitting an immobile plane almost simultaneously. Application and extension of the theory of common invariant cones of linear operators, as well as a blend of analytical and numerical tools lead us to various conditions of complete chatter in terms of physical parameters and initial conditions of the system.

Introduction

Ideally rigid objects establish sustained contact with one another via an infinite sequence of collisions accumulating in finite time. This phenomenon is commonly referred to as complete chatter (CC) or Zeno behaviour. CC can be observed in various mechanical systems including for example a bouncing ball, a rocking block or a rod dropped to the ground. Gravity is crucial for the first two examples, nevertheless, a rod may undergo complete chatter in the absence of external forces. This situation was previously analysed by Sullivan et. al [1, 2], who determined the total number of collisions until separation and developed exact conditions of CC. We revisit their work in the light of the theory of invariant cones of linear operators (or Perron-Frobenius theory). Then we introduce more general versions of this theory in order to analyse the more complex chattering motion of three-dimensional objects.

Problem statement and assumptions

We investigate the motion of a rigid body $\mathcal{B}$ with $n \geq 2$ potential contact points hitting a flat plane $\mathcal{P}$ almost simultaneously, which gives rise to a rapid sequence of collisions. We assume zero friction and focus on objects with n-fold rotational symmetry as in Fig. 1(a). The velocity of the object between two impacts is approximated by constant values and the theory of infinitesimal rotations is used: the orientation of $\mathcal{B}$ is represented by a rotation vector ϕ whose time derivative equals the angular velocity: $\omega = d\phi/dt + O(|\phi|^2)$. The dynamics of the tangential components of the angular velocity (ω_x and ω_y) and the normal component of the translational velocity of the centre of mass (v) are then independent of other velocity components, which are not examined. (Fig. 1(a)). We focus on a projection of the dynamics to velocity space $\mathbb{V}$ spanned by $p = (\omega_x, \omega_w, v)$.

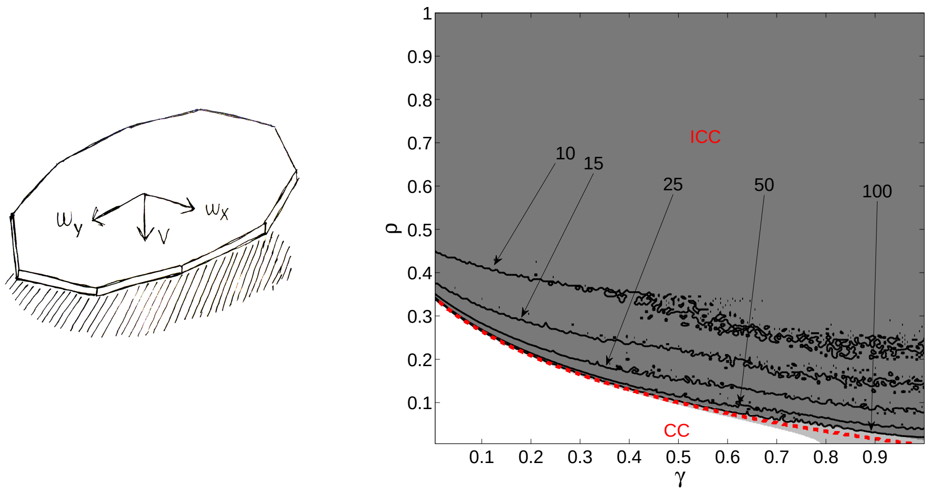

Figure 1: (a) Polyhedral object with n-fold rotation symmetry and the relevant velocity components. The objects examined by us are not necessary flat. (b) Illustration of the results in the case of $n = 5$ as functions of the normalized radius of gyration ρ and the coefficient of restitution γ. The dashed curve represents the boundary of the region where the eigenvalue condition (iii) of Conjecture 1 is satisfied. Enumerated solid lines represent the level curves of the total number of impacts before separation in a direct simulation with randomized initial conditions. The level curves for large numbers accumulate at the boundary between CC and ICC behaviour. Background colors encode the results of a numerical cone finding algorithm: (white = effectively invariant cone found; dark = such a cone does not exist; light grey = the algorithm was inconclusive).

We seek the conditions of two possible outcomes of chattering motion. Complete chatter (CC) happens when $\mathcal{B}$ ends up with $p = 0$ in contact with $\mathcal{P}$. Incomplete chatter (ICC) happens when $\mathcal{B}$ separates from $\mathcal{P}$ after a finite number of impacts. If $n \geq 3$, then we also need to consider a third scenario termed as partial complete chatter (PCC): two vertices may establish sustained contact via an infinite sequence of impacts while the rest of the object keeps moving. This scenario eventually leads to simultaneous impacts at all vertices, which is excluded from the analysis due to lack of a good impact model. Nevertheless we are able to out-rule the possibility of PCC in almost all cases except for a small range of model parameters if $n = 3$.

The case of a rod and invariant cones

With two possible contact points and the observation that a vertex must move away from $\mathcal{P}$ after it has collided with it, the order of colliding vertices is uniquely determined after the first collision: the two endpoints must hit $\mathcal{P}$ in alternating order. The generalized velocity after k impacts can be expressed as $p^{(k)} = (UP)^k p^{(0)}$ where $p^{(0)}$ is the initial velocity, U and P are an impact map and a permutation matrix, respectively.
The authors of [1, 2] show that CC is possible if and only if UP has a positive, real dominant eigenvalue. According to Perron-Frobenius theory [3], this condition is equivalent of UP having an *invariant cone* $\mathcal{K}$, which is mapped by UP into its interior, i.e. for which $UP(\mathcal{K}) \subseteq \mathcal{K}$. This equivalence is not surprising, since – as we show – the rod problem has invariant cones for which $p \in \mathcal{K}$ means that $\mathcal{B}$ approaches $\mathcal{P}$. The persistence of this feature excludes separation and enforces CC.

The case of 3D bodies: common invariant cones

With more than 2 contact points, our velocity-based approach allows an infinite number of possible collision sequences. Indeed, numerical simulation shows complex patterns of impacts, which depend sensitively on parameters and initial conditions. We can now express the velocity after k impacts as

$$p^{(k)} = UP_{i_k} UP_{i_{k-1}} ... UP_{i_1} p^{(0)}$$

where U is again an impact matrix, whereas P_{i_j} are unknown elements of a known set $\{P_1, P_2, ..., P_{n-1}\}$ of $n-1$ rotation matrices. Hence, our model takes the form of a discrete, non-deterministic dynamical system. In analogy with the rod problem, it is a sufficient condition of CC that the set of operators $\mathcal{U} = \{UP_i : \quad i = 1, 2...n-1\}$ has a *common invariant cone* satisfying $UP_i(\mathcal{K}) \subseteq \mathcal{K}$ for any $i = 1, 2...n - 1$. Unfortunately, we found that this sufficient condition is overly restrictive, moreover the existence of common invariant cones is algorithmically undecidable [4].
To fix the first problem, we identify constraints of collision sequences. Specifically, we find a set of $n - 1$ cones $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{n-1}\}$ in velocity space with the property that a collision corresponding to mapping UP_i is impossible unless the preimpact velocity satisfies $p \in \mathcal{C}_i$. We then introduce the concept of *effectively invariant cones*. A cone $\mathcal{K}$ is *effectively invariant* with respect to $\mathcal{U}$ and $\mathcal{C}$ if $U_0 P_i(\mathcal{C}_i \cap \mathcal{K}) \subseteq \mathcal{K}$ for any $i = 1, 2...n-1$. The existence of an effectively invariant cone is a less restrictive sufficient condition of CC.
The second problem is overcome by the development of a numerical algorithm searching for effectively invariant cones as well as a semi-analytical construction in the case of $n = 4$. The results of the cone finding algorithms are compared in our analysis with direct simulation of chattering motion and with eigenvalue analysis of the elements of $\mathcal{U}$ (Fig. 1(b)). Our investigation leads to the following conjecture:

Conjecture 1 *The following three statements are equivalent:*
(i) The object $\mathcal{B}$ undergoes CC for appropriately chosen initial conditions.
(ii) The matrices $\mathcal{U}$ and cones $\mathcal{C}$ have an effectively invariant cone.
(iii) the element UP_1 of $\mathcal{U}$ has a real and positive dominant eigenvalue.

The relation $(ii) \implies (i)$ is straightforward and in the case of $n = 4$, we can also prove analytically that (iii) implies (ii). Nevertheless other aspects of the conjecture look surprising. We have no intuitive argument explaining the observation $(i) \implies (ii)$, i.e. that the existence of an effectively invariant cone is an exact condition of CC. Moreover despite the difficulties associated with the existence of common invariant cones in general, it appears that the question of CC can be reduced to a simple eigenvalue problem as described in (iii).

Conclusions

The chattering of three dimensional objects appears to be a complex phenomenon because the locations of impacts involved in the sequence show complex patterns and depend sensitively on initial conditions and model parameters. Given these difficulties, we used a non-deterministic model of the motion. Surprisingly, we found numerical evidence of simple exact conditions of complete chatter in terms of model parameters. Some of our conjectures were also proven analytically.

References

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