Spectrum of an impact oscillator via nonsmooth modal analysis

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Summary. Recently, the minimal expressions governing the periodic solutions of any autonomous undamped piecewise-linear system with one single purely elastic impact have been derived for any number of degrees of freedom and number of impacts per period (ipp). In the present work, these results are used to exhibit the spectrum of an elementary two-degree-of-freedom vibro-impact oscillator for one, two, and three ipp. The relation between various branches in the Frequency–Energy diagram are discussed. Periodic motions with increasing ipp are shown to converge towards periodic motions with non-zero contact duration. Stability is left for future work.

Introduction

Modal analysis through backbone curves in Frequency–Energy diagrams are becoming familiar in industry for smooth nonlinear systems [2] but there is no systematic way of computing them for nonsmooth systems, involving unilateral constraints on displacement as well as velocity discontinuities. The recent work [1] opens doors to the systematic investigation of the spectrum of a particular class of nonsmooth systems: oscillators with linear free-flights and exhibiting a single unilateral contact condition described with a purely elastic impact law. Any such system can be investigated using the methodology described here, illustrated on a system chosen as simple as possible, see Fig. 1.

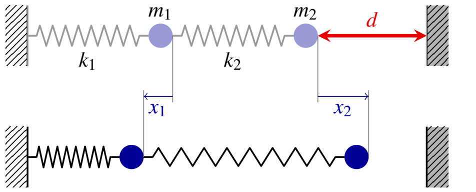


Figure 1: Investigated two-dof spring-mass system with a unilateral contact condition on mass 2 complemented with a purely elastic impact law.

Backbone curves correspond to periodic autonomous solutions of the dynamics. In other words, the objective is to compute the periodic solutions of the system shown in Fig. 1. The orbits are classified by their number of impacts per period (ipp) $k \in \mathbb{N}^\star$. Such solutions can be computed using the framework introduced in [1], briefly summarized below. With no loss of generality, we assume that contact is activated at $t = 0$, that is $x_2(0) = d$; the k successive free-flight durations are denoted by $\sigma_1, \dots, \sigma_k$. The period of the target periodic motion is denoted by $T = \sigma_1 + \cdots + \sigma_k$.

A. A sequence $(\sigma_1, \dots, \sigma_k)$ determines a set of initial conditions from which emanates a periodic solution, except for some isolated values of the k-tuple.
B. For any $k \in \mathbb{N}^*$, there are $k-1$ minimal equations governing the existence of periodic solutions, and these equations are known.
C. The sequences $(\sigma_1, \dots, \sigma_k)$ are generically the union of curves of $\mathbb{R}_+^k$, determined by the $k-1$ equations of B. Therefore, periodic solutions form one-dimensional continua or, in other words, a periodic solution is generically a closed orbit of a two-dimensional manifold in the phase space.

The total energy is determined by the initial conditions and therefore, using A., by the sequence of free-flight durations. From A. and C., finding periodic solutions is reduced to finding curves in the space $(\sigma_1, \dots, \sigma_k)$. Each such curve corresponds to a branch in the Frequency–Energy diagram, where the energy is computed from the initial conditions determined by $(\sigma_1, \dots, \sigma_k)$ (see A.). The corresponding frequency and pulsation are $1/T$ and $2\pi/T$. The curves can be computed via numerical continuation for initial solutions, themselves obtained by running a root-finder algorithm initialized on a Cartesian grid of $\mathbb{R}^k$.

Spectrum for 1, 2, and 3 impacts per period

The spectrum for $k = 1, 2, 3$ is represented in Fig. 2. It becomes very dense as the frequency decreases. Low frequency

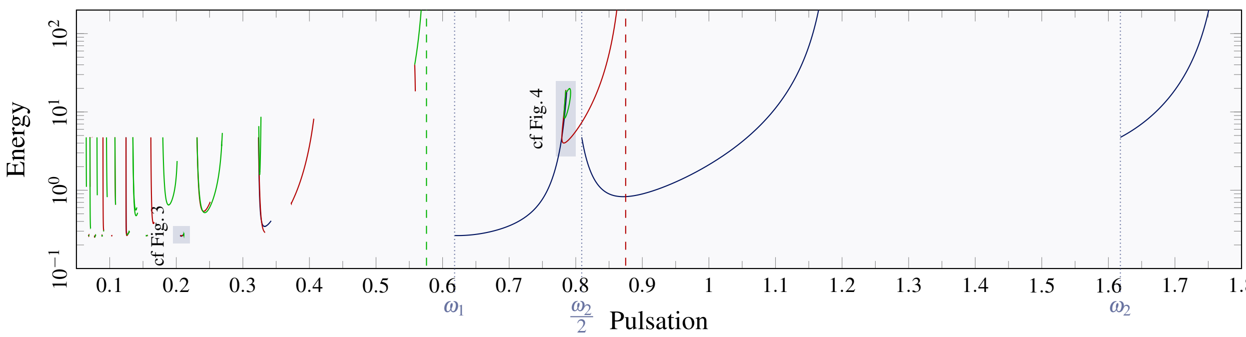


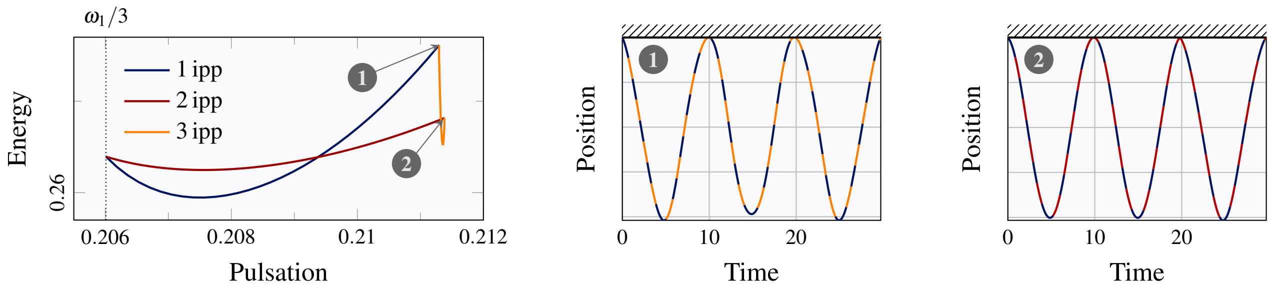
Figure 2: Frequency–Energy diagram for one, two and three impacts per period. $k = 1$ [—]. $k = 2$ [—]. $k = 3$ [—]. The two gray rectangles are detailed in Figs 3 and 4. [- -] and [- -]: Vertical asymptotes.

range corresponds to long periods, for which periodic solutions may only exist if k is sufficiently large. Conversely, the sole periodic solutions in the neighbourhood of the linear natural pulsations ω_1 and ω_2 correspond to $k = 1$, except for the very specific branches converging to a 1 sticking per period (spp), see next section. Each branch terminates as follows:

- Linear mode (a linear natural frequency or one of its harmonics); indeed, many nonsmooth modes of vibration (NSM) emanate from linear modes.
- Divergent with unbounded energy.
- One of the two possible types of grazing:
  1. The second mass grazes the obstacle during a free flight, giving rise to a NSM with a higher number of ipp.
  2. The velocity discontinuity at one time of impact vanishes and gives rise to a NSM of lower k.
- The sequence $(\sigma_1, \ldots, \sigma_k)$ is one of the boundary of the canonical domain, defined as a minimal domain sufficient to generate all solutions using the invariances. For example, when $k = 2$, it suffices to investigate the solutions for $\sigma_1 > \sigma_2$ since the boundary $\sigma_1 = \sigma_2$ corresponds to 1 ipp considered on two periods.

Two interesting phenomena: Loop and Convergence to sticking mode

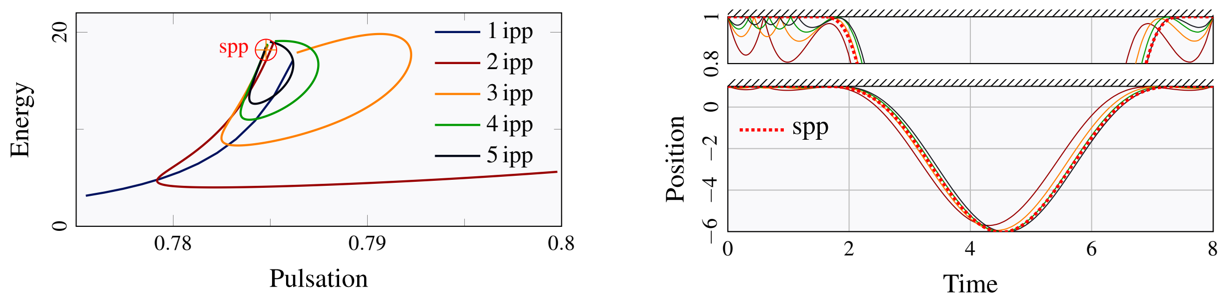
As a consequence of the previous observations, NSM are sometimes arranged in loops, as illustrated in Fig. 3. Starting from the third harmonic of the first linear mode, it is possible to continuously travel along a NSM with $k = 1$, eventually grazing and branching to a NSM with $k = 3$, eventually grazing and branching to a NSM with $k = 2$, looping back to the third harmonic. It is not yet known whether all branches which are not connected to a diverging one are part of a loop. The third possibility would be that some continua correspond to a closed curve in terms of $(\sigma_1, \ldots, \sigma_k)$.



(a) Frequency–Energy diagram **(b)** Junction between 1 and 3 ipp **(c)** Junction between 3 and 2 ipp

Figure 3: $(1, 3, 2)$-loop of the spectrum.

Another interesting feature is the conjectural convergence along k to a "sticking mode" [3], *i.e.* a periodic motion with a non-zero duration of contact, as illustrated in Fig 4.



(a) Frequency–Energy diagram **(b)** Trajectory of the second mass (top: close-up)

Figure 4: Convergence to a "sticking" periodic solution, as the number of impacts per period increases.

Conclusion

The spectrum of a simple impact oscillator has been shown for branches of one, two, and three impacts per period using recent developments [1]. Each end of the branches is either connected to a mode, to another branch or diverges. A loop composed of branches with different number of ipp was observed; a particular point spp seems to attract branches in its vicinity. The next step includes investigating stability to select only branches which might have industrial application.

References

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[3] Huong Le Thi, Stéphane Junca, and Mathias Legrand (2016) Periodic solutions of a two-degree-of-freedom autonomous vibro-impact oscillator with sticking phases, *preprint*, [hal-01305719]