The driven Rayleigh-van der Pol oscillator

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Summary. Synchronization of oscillatory systems is a phenomenon acting from quantum to celestial scale in nature. It is supposed that the controlled synchronization also can be used for applications of active vibration absorption. For this purpose a self-sustained oscillator must be developed that has stable harmonic solutions even when the oscillator is driven by an external harmonic force. This contribution describes how the parameters for a special nonlinear oscillator are obtained, depending on a given harmonic driving force, in such a way that the stable steady-state response is harmonic.

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This contribution is a preliminary work for the analysis if the phenomenon of synchronization of oscillators can be used for applications of active vibration absorption. For the vibration absorption of a system with harmonic vibration behavior it is important that the forced vibration absorber has harmonic steady-state response. The most known self-sustained oscillators have non-harmonic periodic steady-state solutions due to external harmonic forces. In the following the parameters for a special nonlinear oscillator are derived for a given driving force in such a manner that the synchronized steady-state oscillation is harmonic. The entrainment of a self-sustained oscillator by an external force can be seen as the simplest case of synchronization [1].
Considered is a spring-mass system (mass m, stiffness coefficient c) with a controlled force element $F_1(x,\dot{x})$ and external excitation $F_2(t)=\hat{F}\sin(\varOmega t+\alpha)$, according to Figure 1.

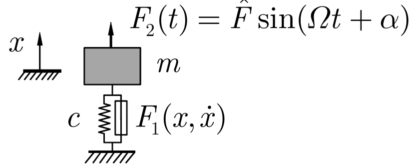


Figure 1: Spring-mass system with controlled force-element and external force.

The first step is the development of an unforced oscillator with an asymptotically stable limit cycle. The equation of motion of the system reads

$$\ddot{x}+\omega^2x=\frac{F_1(x,\dot{x})}{m}+\frac{\hat{F}}{m}\sin(\varOmega t+\alpha),\quad \omega=\sqrt{\frac{c}{m}}. \tag{1}$$

For asymptotically stable periodic motions of the unforced system, thus $\hat{F}=0$, a control scheme for the force element is obtained using speed-gradient method [2]. By defining the goal function

$$Q=\frac{1}{2}(H(x,\dot{x})-H_0)^2 \tag{2}$$

with the total energy of the system $H(x,\dot{x})=\frac{1}{2}m\dot{x}^2+\frac{1}{2}cx^2$ and the desired constant energy $H_0>0$ the control scheme

$$F_1(x,\dot{x})\equiv-\gamma\frac{\partial\dot{Q}}{\partial F_1}=-\gamma(H(x,\dot{x})-H_0)\dot{x}=-\frac{\gamma}{2}(m\dot{x}^2+cx^2-2H_0)\dot{x} \tag{3}$$

with the gain parameter $\gamma>0$ can be obtained. Introducing the control force (3) into the equation of motion (1) with $\hat{F}=0$ and rearranging yields the differential equation of an oscillator,

$$\ddot{x}+\frac{\gamma}{2}\left(\dot{x}^2+\omega^2x^2-\frac{2H_0}{m}\right)\dot{x}+\omega^2x=0. \tag{4}$$

By defining the dimensionless time $\tau=\omega t$ with the notation $'\equiv\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}$ and the dimensionless displacement $q=\frac{x}{\ell}$ with $\ell^2=\frac{2H_0}{m\omega^2}$ the standard form of (4) is given by

$$q''+\varepsilon\left(q'^2+q^2-1\right)q'+q=0,\quad \varepsilon=\frac{\gamma H_0}{m\omega}. \tag{5}$$

The differential equation (5) with the exact harmonic steady state solution $q(\tau)=\sin(\tau-\tau_0)$ [3] is a special case of the general non-harmonic Rayleigh-van der Pol (RvdP) equation, see also [4, 5, 6],

$$q''+\varepsilon\left(\mu q'^2+\nu q^2-1\right)q'+q=0,\quad \mu+\nu>0. \tag{6}$$

Figure 2 shows the phase plot of the harmonic oscillator (5) for several initial conditions, where the steady state solution becomes a circle with radius $\hat{q}=1$.

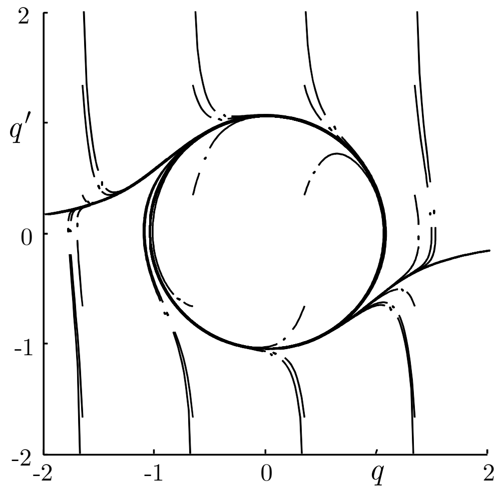


Figure 2: Phase plot of the autonomous harmonic RvdP oscillator from (5) for several initial conditions.

The driven Rayleigh-van der Pol oscillator

Now the forced spring-mass system with an external harmonic excitation force $F_2(t) = \hat{F}\sin(\varOmega t + \alpha)$ with $\varOmega = \eta\omega$ is considered, thus the equation of motion becomes the equation of the harmonic Rayleigh-van der Pol oscillator (5) with an additional driving force,

$$q'' + \varepsilon\left(q'^2 + q^2 - 1\right)q' + q = \bar{\hat{F}}\sin(\eta\tau + \alpha), \quad \bar{\hat{F}} = \frac{\hat{F}}{m\ell\omega^2}. \tag{7}$$

The fundamental angular frequency of the driven oscillator can become a multiple of the driving angular frequency, $\frac{k}{p}\varOmega$, with the whole numbers k, p called synchronization of order k/p. Here the synchronization of order $1/1$ is considered only.

In this section it will be studied under which conditions the steady-state response of the RvdP oscillator due to a harmonic excitation force is harmonic as well. In a first step the existence and stability conditions of the synchronization are derived by solving the question: which excitation force is needed to drive the steady-state solution of the autonomous oscillator (5) with given parameters to a desired steady-state response. The second step is to adjust the parameters of the oscillator for a given excitation force in such a way that these existence and stability conditions are fulfilled.

Driving of the oscillator amplitude

In the phase plane the steady-state solution of the autonomous oscillator (5) becomes the inner circle in Figure 3a with radius $\hat{q} = 1$. The desired steady-state response of the oscillator where only the amplitude is driven by a still unknown excitation force is $\hat{q} = n$ and $\eta = 1$. The desired motion $q(\tau) = n\sin\tau$, corresponding to the outer circle in Figure 3a, is as well the steady-state solution of the autonomous oscillator, described by

$$q'' + \varepsilon\left(q'^2 + q^2 - n^2\right)q' + q = 0. \tag{8}$$

Now the question is what force is needed to drive the oscillator from the inner circle to the outer circle in Figure 3a. To answer this question the equation of the oscillator in (8) is partitioned into

$$q'' + \varepsilon\left(q'^2 + q^2 - 1\right)q' + q = \varepsilon(n^2 - 1)q'. \tag{9}$$

Introducing the steady-state solution $q(\tau) = n\sin\tau$ into the right hand side of (9) yields the equation for the harmonically driven harmonic RvdP oscillator from (5),

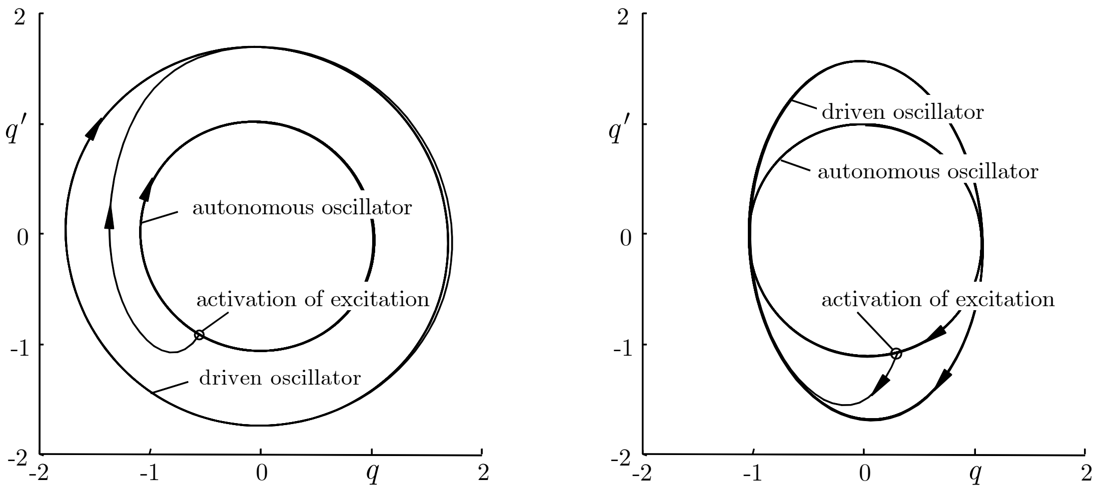
$$q'' + \varepsilon\left(q'^2 + q^2 - 1\right)q' + q = \varepsilon(n^2 - 1)n\cos\tau. \tag{10}$$

Since the steady-state solution $q(\tau) = n\sin\tau$ fulfills equation (10) exactly, the existence condition of the synchronization between the excitation force $\bar{F}(\tau) = \varepsilon(n^2 - 1)n\cos\tau$ and the oscillator is fulfilled. To determine if the synchronization is stable the equation (10) is harmonically linearized [7] at the dimensionless angular frequency $\eta = 1$ into

$$q'' + 2\delta q' + q = \varepsilon(n^2 - 1)n\cos\tau, \tag{11}$$

with

$$2\delta = \frac{\varepsilon}{\pi}\int_0^{2\pi}\left(n^2\cos^2\tau + n^2\sin^2\tau - 1\right)n\cos\tau\frac{\cos\tau}{n}\,\mathrm{d}\tau = \varepsilon(n^2 - 1). \tag{12}$$



(a) (b)

Figure 3: Phase plots of steady-state solutions. (a) Driving of the oscillator amplitude from $q(\tau) = \sin\tau$ to $q(\tau) = n\sin\tau$ (b) Driving of the oscillator phase angle from $q(\tau) = \sin\tau$ to $q(\tau) = \sin\eta\tau$.

Thus the synchronization of order $1/1$ is stable for

$$\delta = \frac{\varepsilon(n^2-1)}{2} > 0, \tag{13}$$

and thus for the amplitude $|n| > 1$. Now the results are used to determine the amplitude response n of the oscillator (7) due to an given excitation force $\bar{F}(\tau) = \bar{\hat{F}}\sin(\tau+\alpha) = \bar{\hat{F}}\cos\alpha\sin\tau + \bar{\hat{F}}\sin\alpha\cos\tau$. The comparison of the given force with the right hand side of (10),

$$\varepsilon(n^2-1)n\cos\tau = \bar{\hat{F}}\cos\alpha\sin\tau + \bar{\hat{F}}\sin\alpha\cos\tau, \quad \bar{\hat{F}} = \frac{\hat{F}}{m\ell\omega^2} \tag{14}$$

shows that the amplitude of the oscillator can only be driven under the mentioned assumptions for the phase angles

$$\alpha_{1/2} = \pm\frac{\pi}{2}. \tag{15}$$

By introducing the two possible solutions (15) into (14) the conditions between the dimensionless force amplitude $\bar{\hat{F}}$ and the amplitude of the oscillator n can be determined

$$\varepsilon(n^2-1)n = \pm\bar{\hat{F}}. \tag{16}$$

Equation (16) contains two cubic equations for the unknown amplitude n with the six solutions

$$n_{1/2} = \frac{1}{3\,\sigma_{1/2}^{\frac{1}{3}}} + \sigma_{1/2}^{\frac{1}{3}}, \quad n_{3/4,5/6} = \frac{\sqrt{3}\,\sigma_{1/2}^{\frac{1}{3}}\,\mathrm{i}}{2} - \frac{1}{6\,\sigma_{1/2}^{\frac{1}{3}}} - \frac{\sigma_{1/2}^{\frac{1}{3}}}{2} \pm \frac{\sqrt{3}\,\mathrm{i}}{6\,\sigma_{1/2}^{\frac{1}{3}}} \tag{17}$$

with

$$\sigma_{1/2} = \sqrt{\frac{\bar{\hat{F}}^2}{4\,\varepsilon^2} - \frac{1}{27}} \pm \frac{\bar{\hat{F}}}{2\,\varepsilon}. \tag{18}$$

Introducing (17) and (16) into the stability condition (13) yields the stability conditions

$$\varepsilon(n_i^2-1) = \begin{cases} \dfrac{\bar{\hat{F}}}{n_i} > 0, & i = 1,3,5, \\[2ex] -\dfrac{\bar{\hat{F}}}{n_i} > 0, & i = 2,4,6. \end{cases} \tag{19}$$

The two equations in (16) have for each time instant one real solution in case of $\bar{\hat{F}} > \varepsilon\sqrt{\frac{4}{27}}$, namely n_1 and n_4, and each time three real solutions in case of $\bar{\hat{F}} < \varepsilon\sqrt{\frac{4}{27}}$. It can be shown that only solutions n_1 (for $\alpha = \frac{\pi}{2}$) and n_4 (for $\alpha = -\frac{\pi}{2}$) are real solutions of (16) fulfilling the stability conditions (19) for arbitrary values $\bar{\hat{F}} > 0$.

Driving of the oscillator phase angle
Now the question arises what force is needed to drive only the phase angle of the autonomous oscillator (5) thus the steady-state response is given by $q(\tau) = \sin\eta\tau$, see also Figure 3b. This steady-state response is as well the solution of the autonomous oscillator described by

$$q'' + \varepsilon\left(q'^2 + \eta^2 q^2 - \eta^2\right) q' + \eta^2 q = 0. \tag{20}$$

Partitioning of (20) into

$$q'' + \varepsilon\left(q'^2 + q^2 - 1\right) q' + q = (1-\eta^2)q^2 q' + (\eta^2 - 1)q' + (1-\eta^2)q \tag{21}$$

and introducing the steady-state solution $q(\tau) = \sin\eta\tau$ into the right hand side yields the equation for the driven harmonic RvdP oscillator from (5),

$$q'' + \varepsilon\left(q'^2 + q^2 - 1\right) q' + q = (1-\eta^2)\eta\sin^2\eta\tau\cos\eta\tau + (\eta^2-1)\eta\cos\eta\tau + (1-\eta^2)\sin\eta\tau. \tag{22}$$

Obviously the excitation force in (22) is periodic but not harmonic for $\eta \neq 1$ as desired. As a result the harmonic RvdP from (5) driven by a harmonic force $\hat{F}\sin(\Omega t + \alpha)$ has non-harmonic steady-state response in general, exemplary seen in Figure 4a, except of the case $\Omega = \omega, \alpha = \pm\frac{\pi}{2}$. In the following it is studied if the parameters of the non-harmonic RvdP oscillator from (6) can be adjusted for a given harmonic excitation force in such a way that the steady-state response is harmonic as well.

The driven non-harmonic Rayleigh-van der Pol oscillator
In the following it is studied which force is needed to drive the non-harmonic RvdP oscillator with given parameters from (6) to a desired harmonic response, see also Figure 4b. After solving the existence and stability conditions the parameters of the oscillator are adjusted for a given excitation force $\bar{F}(\tau) = \hat{\bar{F}}\sin(\eta\tau + \alpha)$.
The desired steady-state response $q(\tau) = n\sin\eta\tau$ is as well the steady-state solution of the autonomous oscillator described by

$$q'' + \varepsilon\left(q'^2 + \eta^2 q^2 - \eta^2 n^2\right) q' + \eta^2 q = 0. \tag{23}$$

Partitioning of (23) yields for arbitrary constant values of a, b,

$$q'' + \varepsilon\left(q'^2 + \eta^2 q^2 - \eta^2 n^2 + \frac{b}{\varepsilon n\eta}\right) q' + \left(\eta^2 + \frac{a}{n}\right) q = \frac{a}{n} q + \frac{b}{n\eta} q'. \tag{24}$$

Introducing the harmonic solution $q(\tau) = n\sin\eta\tau$ into the right hand side of (24) the inhomogeneous equation

$$q'' + \varepsilon\left(q'^2 + \eta^2 q^2 - \eta^2 n^2 + \frac{b}{\varepsilon n\eta}\right) q' + \left(\eta^2 + \frac{a}{n}\right) q = a\sin\eta\tau + b\cos\eta\tau. \tag{25}$$

leads to a driven non-harmonic oscillator.
Remark: The equation of an general RvdP oscillator $q'' + \varepsilon\left(q'^2 + \nu q^2 - \xi\right) q' + \chi^2 q = 0$ is equivalent to the standard form (6) due to a coordinate transformation.
Due to the excitation force $\bar{F}(\tau) = a\sin\eta\tau + b\cos\eta\tau$ the synchronized steady-state response $q(\tau) = n\sin\eta\tau$ is existing, but the stability has to be studied in the following. To this end equation (25) is harmonically linearized at the dimensionless angular frequency η into

$$q'' + 2\delta q' + \left(\eta^2 + \frac{a}{n}\right) q = a\sin\eta\tau + b\cos\eta\tau, \tag{26}$$

with

$$2\delta = \frac{\varepsilon}{\pi}\int\limits_0^{\frac{2\pi}{\eta}} \left(\overbrace{n^2\eta^2\cos^2\eta\tau + \eta^2 n^2\sin^2\eta\tau - \eta^2 n^2}^{0} + \frac{b}{\varepsilon n\eta}\right) n\eta\cos\eta\tau \frac{\cos\eta\tau}{n}\,\mathrm{d}\tau = \frac{b}{n\eta}. \tag{27}$$

The synchronization of order $1/1$ is stable for

$$\delta = \frac{b}{2n\eta} > 0. \tag{28}$$

Now the parameters of the oscillator are adjustet to a given harmonic excitation force. If the nonharmonic RvdP oszillator in (25) will be excited by the external force $\bar{F}(\tau) = \hat{\bar{F}}\sin(\eta\tau + \alpha) = \hat{\bar{F}}\cos\alpha\sin\eta\tau + \hat{\bar{F}}\sin\alpha\cos\eta\tau$ with the arbitrary dimensionless force amplitude $\hat{\bar{F}}$ and the arbitrary phase angle α the synchronized harmonic steady-state response exists if the condition

$$a\sin\eta\tau + b\cos\eta\tau = \hat{\bar{F}}\cos\alpha\sin\eta\tau + \hat{\bar{F}}\sin\alpha\cos\eta\tau, \tag{29}$$

and by this

$$a = \hat{\bar{F}}\cos\alpha, \quad b = \hat{\bar{F}}\sin\alpha \tag{30}$$

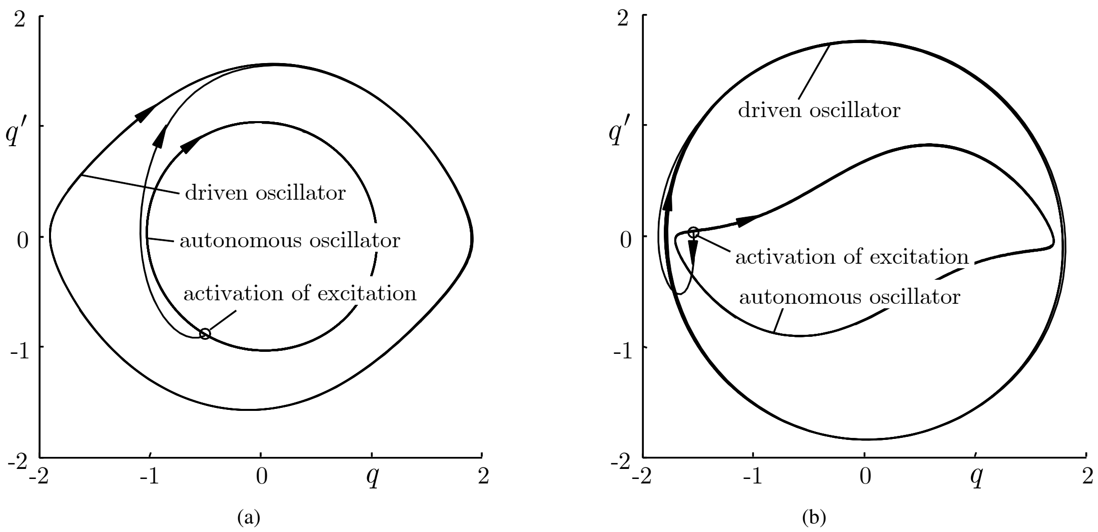


Figure 4: Phase plots. (a) Harmonical driven harmonic RvdP oscillator with $\eta \neq 1$. (b) Harmonical driven general RvdP oscillator with the steady-state solution $q(\tau) = n \sin \eta\tau$.

is fulfilled. By introducing the results (30) into (25) the equation of the non-harmonic oscillator driven by a given excitation force can be obtained

$$q'' + \varepsilon \left(q'^2 + \eta^2 q^2 - \eta^2 n^2 + \frac{\bar{\hat{F}} \sin\alpha}{\varepsilon n \eta} \right) q' + \left(\eta^2 + \frac{\bar{\hat{F}} \cos\alpha}{n} \right) q = \bar{\hat{F}} \sin(\eta\tau + \alpha). \tag{31}$$

Remark: The same results can be obtained by introducing the ansatz $q(\tau) = n \sin \eta\tau$ into $q'' + \varepsilon \left(q'^2 + \nu q^2 - \xi \right) q' + \chi^2 q - \bar{\hat{F}} \sin(\eta\tau + \alpha) = 0$. From the requirement that all coefficients of the harmonic terms are vanishing a unique solution for the parameters ν, ξ, χ can be found.

Back-transformation to the original coordinates

The back-transformation of (31) to the original coordinates of system (1) with $q = \frac{x}{\ell}, q' = \frac{\dot{x}}{\omega\ell}, q'' = \frac{\ddot{x}}{\omega^2\ell}$ and $\ell^2 = \frac{2H_0}{m\omega^2}$ yields

$$\ddot{x} + \frac{\gamma}{2} \left(\dot{x} + \Omega^2 x^2 - \left(\frac{2\eta^2 n^2 H_0}{m} - \frac{2\hat{F} \sin\alpha}{\gamma n m \Omega} \sqrt{\frac{m\omega^2}{2H_0}} \right) \right) \dot{x} + \left(\Omega^2 + \frac{\hat{F} \cos\alpha}{n\sqrt{2mH_0}} \right) x = \frac{\hat{F}}{m} \sin(\Omega t + \alpha). \tag{32}$$

where the desired energy H_0 is a free parameter. The desired energy can be defined reasonably by a desired amplitude response

$$\hat{x} = n\ell = n\sqrt{\frac{2H_0}{m\omega^2}} \quad \rightarrow \quad H_0 = \frac{\hat{x}^2 m\omega^2}{2n^2}. \tag{33}$$

Introducing the desired energy from (33) into (32) yields the equation for the non-harmonic oscillator with adjusted parameters for the harmonic excitation force,

$$\ddot{x} + \frac{\gamma}{2} \left(\dot{x}^2 + \Omega^2 x^2 - \left(\Omega^2 \hat{x}^2 - \frac{2\hat{F} \sin\alpha}{\Omega \hat{x} \gamma m} \right) \right) \dot{x} + \left(\Omega^2 + \frac{\hat{F} \cos\alpha}{m\hat{x}} \right) x = \frac{\hat{F}}{m} \sin(\Omega t + \alpha). \tag{34}$$

The back-transformation of the stability condition (28) yields

$$\frac{b}{2n\eta} = \frac{\hat{F} \sin\alpha}{2m\hat{x}\Omega} > 0. \tag{35}$$

From the requirement that the angular eigenfrequency of the harmonically linearized system (26) must be positive for stable solutions, the additional condition

$$\Omega^2 + \frac{\hat{F} \cos\alpha}{m\hat{x}} > 0 \tag{36}$$

must be satisfied for stability of the original system.

The searched state dependent control force from (1) then becomes

$$F_1(x, \dot{x}) = -\frac{\gamma}{2}\left(\dot{x} + \Omega^2 x^2 - \left(\Omega^2 \hat{x}^2 - \frac{2\hat{F}\sin\alpha}{\Omega\hat{x}\gamma m}\right)\right)\dot{x} - \left(\Omega^2 - \omega^2 + \frac{\hat{F}\cos\alpha}{m\hat{x}}\right)x. \tag{37}$$

If system (1) is excited by the force $\hat{F}\sin(\Omega t + \alpha)$, and control law (37) is used for the state dependent force element, then the steady-state response of the system reads $x(t) = \hat{x}\sin\Omega t$ with a desired amplitude $\hat{x}$ if the stability conditions (35) and (36) are fulfilled. Additional numerical simulations confirmed this result. Now the control law (37) can be used for applications of active vibration absorption.

Conclusions

Based on speed gradient method the harmonic Rayleigh-van der Pol oscillator is derived from a spring-mass system with a controlled force element. By using the harmonic steady-state behavior of the unforced oscillator conditions were defined, that are necessary for the harmonic steady-state response of the forced non-harmonic oscillator. From these conditions the parameters for the forced oscillator, depending on a given excitation force, can be derived to ensure stable steady-state harmonic solutions.

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