Vibration power flow analysis of typical nonlinear oscillators

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Summary. Two typical nonlinear oscillators, the Van der Pol and the Duffing oscillators are re-examined from a viewpoint of vibration power flows. The former is a frequently-used model for nonlinear damping with limit cycle oscillation behaviour, while the latter is often used for models with cubic nonlinear stiffness. The power flow analysis approach for nonlinear dynamical systems is firstly presented. This is followed by investigations of the dynamics of these oscillators in terms of vibration power flows. The effects of nonlinearity on time-averaged input and dissipated powers and the maximum kinetic energy are studied. Some intrinsic nonlinear power flow behaviour is revealed. Potential benefits of employing nonlinearities for vibration energy harvesting and vibration mitigation are demonstrated.

The power flow analysis (PFA) approach

The vibration power flow analysis approach has become a wide-accepted to characterise the dynamic behaviour of complex systems and coupled structures. Vibration power flow combines the effects of force and velocity in a single quantity and thus provides a good performance index to quantify the vibration transmission at the interface of sub-systems within an integrated structures. Since the introduction of power flow concept in 1980s, it has been successfully adopted to investigate many linear passive or active vibration control systems [1]. However, it should be pointed out that engineering systems are inherently nonlinear. Many recent studies have also shown that nonlinear elements may be used to improve the performance of vibration isolators and dynamic vibration absorbers to enhance vibration mitigation. It is thus of importance to develop vibration power flow analysis approach for nonlinear dynamical systems so as to understand their associated power flow characteristics [2]. This can also provide new insight to dynamic designs of systems for effective vibration suppressions or energy harvesting purpose [3-6].
To demonstrate vibration power flow formulations, the governing equation of a dynamical system may be written as

$$[\mathbf{M}]\{\ddot{\boldsymbol{x}}\} + [\mathbf{C}]\{\dot{\boldsymbol{x}}\} + [\mathbf{K}]\{\boldsymbol{x}\} = \{\boldsymbol{f}\}, \tag{1}$$

where $[\mathbf{M}]$, $[\mathbf{C}]$ and $[\mathbf{K}]$ are the mass, damping, and stiffness matrices which may not be constant, but change with x and/or $\dot{x}$ due to the nonlinearity of the system, $\{\ddot{\boldsymbol{x}}\}$, $\{\dot{\boldsymbol{x}}\}$ and $\{\boldsymbol{x}\}$ are the acceleration, velocity and displacement vectors, and $\{\boldsymbol{f}\}$ is the external force vector. The equation of power balance may be obtained by pre-multiplying Eq. (1) by the velocity vector $\{\dot{\boldsymbol{x}}\}^{\mathrm{T}}$:

$$\dot{K} + p_d + \dot{U} = p_{in}, \tag{2}$$

where $\dot{K} = \{\dot{\boldsymbol{x}}\}^{\mathrm{T}}[\mathbf{M}]\{\ddot{\boldsymbol{x}}\}$ and $\dot{U} = \{\dot{\boldsymbol{x}}\}^{\mathrm{T}}[\mathbf{K}]\{\boldsymbol{x}\}$ represent the rates of change of the kinetic and the potential energies of the system, respectively, while $p_d = \{\dot{\boldsymbol{x}}\}^{\mathrm{T}}[\mathbf{C}]\{\dot{\boldsymbol{x}}\}$ and $p_{in} = \{\dot{\boldsymbol{x}}\}^{\mathrm{T}}\{\boldsymbol{f}\}$ are the instantaneous dissipated and input powers, respectively. The superscript T denotes transpose matrix. Eq. (2) may be averaged in a time span from $t = t_0$ to $t = t_0 + \Delta t$ to obtain time-averaged power flow equation:

$$\Delta K + \int_{t_0}^{t_0+\Delta t} p_d \mathrm{d}t + \Delta U = \int_{t_0}^{t_0+\Delta t} p_{in} \mathrm{d}t, \tag{3}$$

where ΔK and ΔU are the net changes in kinetic and potential energies, respectively. Eq. (3) may be time-averaged to obtain time-averaged power flow equation. The averaging time may be taken to be the period of a periodic response. There may be different ways of obtaining the power flow behaviour of the systems. One straightforward approach is to transform Eq. (1) into a set of first-order differential equations, which may be solved by direct numerical integrations to obtain the response. The instantaneous and time-averaged power flow variables may then be found. Another approach is to adopt a combination use of the harmonic balance method and numerical methods. This involves expressions of the steady state response and the nonlinearity in Eq. (1) with a number of harmonic terms. Balancing these the corresponding harmonic terms results in nonlinear algebraic equations which can be solved by numerical methods to obtain the displacement / velocity response and subsequently the power flow variables.

Application of PFA to typical nonlinear oscillators

To demonstrate the application of PFA approach for nonlinear dynamical systems, two typical nonlinear systems, the Van der pol oscillator and the Duffing oscillator, are investigated from a power flow perspective so as to examine the effects of damping and stiffness nonlinearities on vibration power flow characteristics. The dynamic equation of the Van der Pol oscillator is expressed by

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = f\cos\omega t, \tag{4}$$

The equations of power banlance is

$$\ddot{x}\dot{x} + \alpha(x^2 - 1)\dot{x}\dot{x} + x\dot{x} = \dot{x}f\cos\omega t, \tag{5}$$

The power flow varibles of the system are obtained by using analytical approximation and also numercial integrations. In the content that follows, some results are presented to show the intrisic nonlinear power flow phenomena of the systems. It is well-known that the unforce Van der Pol oscillator is characterised by a limit cycle oscillator, as shown in Fig. 1(a). The steady-state response evolves to a stable limit cycle for different initial conditions, as illustrated by points B and C in the figure. Correspondingly, when the dissipated power p_d is added as another coordinate, the limit cycle is transformed into a three-dimensional form, as shown in Fig. 1(b). It is found that p_d may become negative.

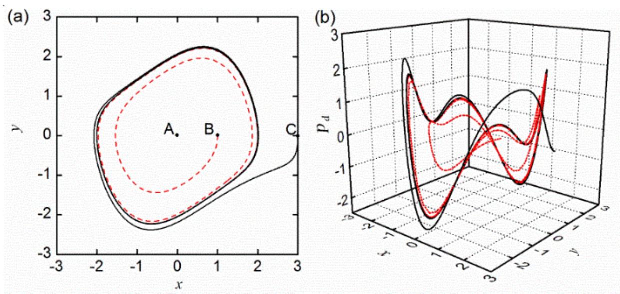

Figure 1. Limit cycle oscillations (a) two-dimensional form and (b) three-dimensional form $(\alpha = 0.5, f = 0)$.

For the forced Van der Pol oscillator, analytical approximation and numerical integrations are used to obtain the time-averaged power flow level. To illustrate, a first-order approximation of the periodic displacement and velocity response is expressed by $x = r_1 \cos(\omega t + \phi), y = -\omega r_1 \sin(\omega t + \phi)$, respectively. Following a harmonic balance approximation, the frequency-response relationship is obtained. An analytical expression of time-averaged input power can then be found. As shown in Fig. 2, the system exhibits quasi-periodic motion in a large range of excitation frequency in the low and the high frequency range and periodic motion between approximately $\omega = 0.54$ and $\omega = 1.28$. Bifurcations occur at these two frequencies due to the change in the type of the response. It shows that the analytical approximation of time-averaged input power agrees quite well with numerical integration results. Other frequency components need to be included for analytical formulation of power flows associated with quasi-periodic responses.

The power flow analysis (PFA) has also been used to investigate the power flow behaviour of the Duffing oscillator, with some results reported previously in ref [3]. The power flow characteristics associated with nonlinear stiffness systems have been explored and employed for nonlinear vibration mitigation [4-6].

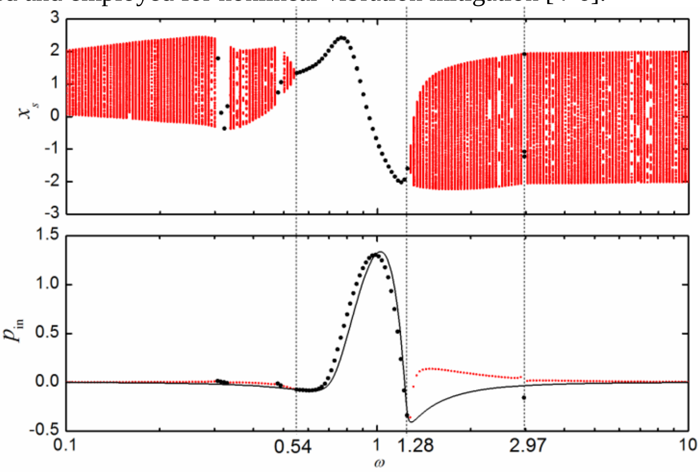

Figure 2. Power flow characteristics of the forced Van der pol oscillator. (a) Bifurcation diagram and (b) time-averaged input power $(\alpha = 0.5, f = 1)$. Black dots: periodic responses; red dots: quasi-periodic responses. Lines: analytical approximations.

Conclusions

Vibration power flow analysis was carried out to re-examined typical nonlinear oscillators from a power flow perspective. It is shown that the dissipated power of the unforced Van der Pol oscillator may be negative. For the forced Van der Pol oscillator, the time-averaged input power may also be negative at some excitation frequencies. This suggests a net energy output from the system. It is of direct contrast to linear systems and may be used for enhancing vibration energy harvesting. Some power flow characteristics of the Duffing oscillator and insights were also obtained to improve vibration control performance by introducing nonlinear stiffness to vibration mitigation systems.

References

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