The Painlevé paradox and blowup - Part I

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Summary. It is well-known that the dynamics near hyperbolic equilibria are described by Hartman-Grobman and that partially hyperbolic equilibria can be studied through center manifolds. The blowup method was developed to enable the use of these methods to situations when initially there are just zero eigenvalues of the linearization. Loosely speaking, the blowup approach blows up the singularity so that the zero eigenvalues of the resulting, transformed vector-field can be divided out. The corresponding division gives rise to a new vector-field, whose orbits agree with the original one away from the singularity, that can then be studied by the hyperbolic methods.

This talk is part I of two parts, where we apply blowup and singular perturbation theory to the regularization of the classical, nonsmooth Painlevé problem. Here we will first demonstrate blowup on a simpler, yet related, toy problem. Subsequently we will then explain the Painlevé problem and its regularization. Finally we will use blowup to prove the existence of canards.

Nonsmooth modelling and singular perturbations

Contact in mechanics is frequently modelled by nonsmooth forces. Coulomb's law of friction is an obvious example: The friction force changes abruptly when passing through zero relative velocity. However, nonsmooth models are not in general mathematically well-posed. The classical example of Painlevé [9], consisting of a slender rod slipping along a rough surface (see Fig. 1), described by Coulomb's friction and an unilateral constraint, is a simple example of a nonsmooth system with both loss of existence and uniqueness of solutions, now known collectively as *Painlevé paradoxes* [1, 2]. Often one resorts to nonsmooth models to hide or idealise phenomena that happen on a small scale and are deemed unimportant. The presence of paradoxes, as in the Painlevé's example, is a failure of the nonsmooth model to provide a complete description. Things that we discard at the microscale become important. Coulomb's law, for example, is known to be an idealization. More accurate models of friction, e.g. [10], are smooth, albeit with sharp transitions. Contact is therefore probably a singular perturbation problem rather than a nonsmooth one.

For a mathematician, singular perturbed models are easier to study than their nonsmooth counterparts. For one thing, we have (local) existence and uniqueness of solutions. But we also have all the theory of smooth dynamical systems at our disposal. For example, Fenichel's geometric theory of singular perturbations [5] and blowup [3, 7]. To demonstrate the potential of these theories in the context of nonsmooth systems, consider the following simple example:

$$\dot{x} = 1, \quad \dot{y} = x\tanh(y\epsilon^{-1}) + \alpha x^2(1+\tanh(y\epsilon^{-1})). \tag{1}$$

The singular limit $\epsilon = 0$ is piecewise smooth

$$\dot{x} = 1, \quad \dot{y} = \begin{cases} x + 2\alpha x^2 & \text{for } y > 0, \\ -x & \text{for } y < 0. \end{cases} \tag{2}$$

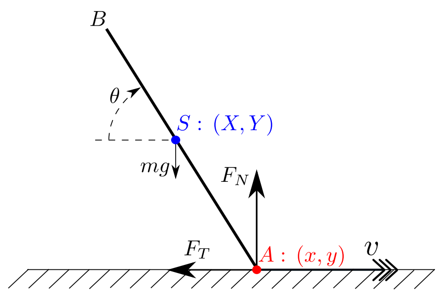

Figure 1: The classical Painlevé problem.

using that $\tanh(z) \to \pm 1$ for $z \to \pm\infty$. The phase portrait of (2) (as a Filippov system) is sketched in Fig. 2 near $(x, y) = (0, 0)$. The orbits in red γ^+ and blue γ^- of $(2)_{y>0}$ and $(2)_{y<0}$, respectively, are both tangent to $y = 0$ at $x = 0$. There is therefore a *fold-fold singularity* p at $(x, y) = 0$. Solutions starting within the shaded region (such as the orbit in black) are not forward unique: There exists infinitely many forward trajectories through the point p. Some candidates are illustrated in the figure. But for (1), we are able to obtain the following:

Theorem 1 *Fix $\alpha \neq 0$ and any initial condition within the shaded region of Fig. 2. Then the forward orbit converges to (a) $\gamma^+ \cap \{x > 0\}$ if $\alpha > 0$ or (b) $\gamma^- \cap \{x > 0\}$ if $\alpha < 0$, for $x > 0$, as $\epsilon \to 0$.* □

To prove the theorem, we first apply the following scaling $y = \epsilon\hat{y}$. In terms of $(x, \hat{y}, \epsilon)$, we then obtain a normally attracting critical set $C_a : \hat{y} = 0,\, x < 0, \epsilon = 0$ and a repelling one $C_r : \hat{y} = 0,\, x > 0,\, \epsilon = 0$. Compact subsets of $C_{a,r}$ perturb by Fenichel's theory to slow manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$, respectively, for $0 < \epsilon \ll 1$. However, there is also a line $p : x = 0,\, \hat{y} \in \mathbb{R},\, \epsilon = 0$ of normally non-hyperbolic critical points. To describe the passage near this line, we follow [3, 7] and blow it up by applying the following transformation:

$$\Phi : x = r\bar{x},\, \epsilon = r^2\bar{\epsilon}, \quad r > 0, \quad (\bar{x}, \bar{\epsilon}) \in S^1 : \bar{x}^2 + \bar{\epsilon}^2 = 1. \tag{3}$$

This transformation blows up the line p to a cylinder $\bar{p} : r = 0,\, (\bar{x}, \bar{\epsilon}) \in S^1,\, \hat{y} \in \mathbb{R}$ and transforms the vector-field on $(x, \hat{y}, \epsilon)$ to a vector-field $\overline{X}$ on $(\hat{y}, r, (\bar{x}, \bar{\epsilon})) \in \mathbb{R} \times \mathbb{R}_+ \times S^1$ by pullback. Here $\overline{X}|_{r=0} = 0$. However, the weights of r in (3), have been chosen so that $\widetilde{X} = r^{-1}\overline{X}$ is well-defined and satisfy $\widetilde{X}|_{r=0} \neq 0$. This *desingularization* does not alter the phase portrait of $\overline{X}$ outside the cylinder $\mathbb{R} \times \{0\} \times S^1$. But since $\widetilde{X}|_{r=0} \neq 0$, we can use perturbation techniques of dynamical systems to perturb away from $r = 0$ and prove the statement of the theorem. (The result is in fact independent of the *regularization* function. For example, a similar statement holds true if tanh is replaced by $\frac{2}{\pi}\arctan$.)

The Painlevé problem

In the second part of this paper, we focus on the Painlevé problem. In Fig. 3, we show the $(\theta, \phi = \dot{\theta})$-phase plane of the rigid body motion with the special point P [4] that divides the configurations of the rod into four quadrants. In the green region, the rod will lift off from contact with the rigid surface. In the yellow region, the rod slips along the surface. Genot and Brogliato [4] showed that for a sufficiently large friction coefficients $\mu > \mu_P$ there is a special slipping orbit (γ^s in Fig. 3) within the yellow quadrant which reaches the point P in finite time. The rigid body formulation is not able to predict what happens beyond this point. Even worse, in the green region the rigid body formulation has no solution, while there exists two solutions within the purple region, corresponding to slipping and lift-off.

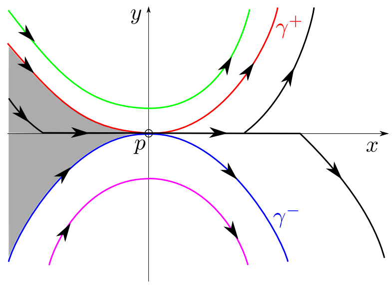

Figure 2: Phase portrait of (2). Solutions are forward non-unique from p.

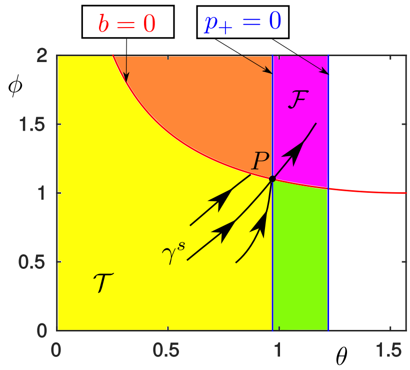

Figure 3: The $(\theta, \phi = \dot{\theta})$-plane for the classical Painlevé problem of Fig. 1.

To resolve the paradoxical situations we assume that there is compliance at the point A between the rod and the surface, when they are in contact (see Fig. 1). Following [8], we assume that there are small excursions into $y < 0$. Then we take the nonnegative normal force $F_N(y, w)$ as a piecewise smooth function of (y, w): $F_N(y, w) = \epsilon^{-1}\left[F(\epsilon^{-1}y, w)\right]$ where $F(\hat{y}, w) = -\hat{y} - \delta w + \mathcal{O}((\hat{y} + w)^2$. Here ϵ is a small parameter related to the spring constant, δ is the damping and the operation $[\cdot]$ is defined by

$$[f(y,w)] \equiv \begin{cases} 0 & \text{for} \quad y > 0, \\ \max\{f(y,w), 0\} & \text{for} \quad y \leq 0. \end{cases} \tag{4}$$

The choice of scaling [8] ensures that the critical damping coefficient is independent of ϵ.

Results

We proceed as in the toy problem (1) above: First we apply the scaling $y = \epsilon^2 \hat{y}$. Then we obtain an attracting critical manifold C_a (of focus-type close to P) as a graph over the yellow region in Fig. 3 and a saddle-type critical manifold C_r as a graph over the purple region. The two manifolds carry a reduced flow coinciding with the slipping dynamics along $y = 0$ described by the rigid body system. However, the closure of the two sets $\overline{C}_{a,r}$ intersect in a line $\widehat{P}$ (the compliant version of P) of non-normally hyperbolic critical points. We blowup this line and obtained the following:

Theorem 2 *[6] Let $\mu > \mu_P$. Then for $0 < \epsilon \ll 1$ there exists a* canard *orbit γ_ϵ^s for the compliant system, which connects the attracting Fenichel slow manifold $S_{a,\epsilon}$ with the stable manifold of the repelling Fenichel slow manifold $S_{r,\epsilon}$. γ_ϵ^s is $o(1)$-close to γ^s and it divides $S_{a,\epsilon}$ into orbits that lift off from those that eventually stick.* □

The proof is not standard as we only gain ellipticity (rather than hyperbolicity) of C_a at the blowup of $\widehat{P}$. Nevertheless, using normal form transformations and an extended version of the center manifold theorem, we are able to reduce the proof of the theorem to asymptotic properties of solutions of the following third order linear equation

$$y^{(3)}(\theta_2) = \theta_2 y'(\theta_2) + (1 - \xi) y(\theta_2).$$

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