Three to One Internal Resonance of Modes With Different Decay Rates

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Summary. We investigate the dynamics of a system with a driven nonlinear mode that is coupled to a linear mode with a frequency ratio of 1:3. The nonlinear driven mode is described by the simplest model that displays amplitude dependence of the vibration frequency (i.e., a Duffing equation) and enables a driving-tuned internal resonance. Both modes are lightly linearly damped, but the decay rate of the linear mode is much faster than the decay rate of its nonlinear counterpart. This allows us to simplify the analysis by assuming that the linear (fast) mode adiabatically follows the nonlinear (slow) mode. Then the dynamics can be described by a single mode with quintic nonlinear damping and stiffness terms that result from the backaction of the linear mode. We show that this backaction can significantly change the dynamical response of the driven mode. Thus, instead of a standard forced Duffing response curve, one obtains an anomalously strong nonlinear friction along with a peculiar response curve that is marked by complex hysteretic behavior. These effects can be thought of as a consequence of the "repulsion" of the vibration frequencies away from the resonance condition. The results of this study bear on the experimental observations in micro- and nano-scale resonators that exhibit internal resonance.

One of the most important and interesting nonlinear phenomena of multi-mode dynamical systems is internal resonance (IR), in which the system vibrational modes (which, by definition, are linearly uncoupled) interact strongly even in the presence of weak nonlinear coupling when the modal frequencies are rationally related ($\omega_2/\omega_1 \approx n/m$). In the conservative case (i.e., in the absence of dissipation), a relatively simple picture emerges. Here, the resonance leads to the onset of nonlinear vibrations accompanied by energy oscillations between the resonating modes, reminiscent of linear resonance between coupled harmonic oscillators. However, on a finer scale, the picture is more complicated as the motion can display a whole range of frequencies, and even dynamical chaos. On the other hand, in the non-conservative case the modes are interacting with the environment, which leads to dissipation and noise, and thus only relatively low-order IRs ($|n, m| \leq 3$) are generally observed in experiments. If the modes have very different decay rates, one of them can serve as a type of thermal reservoir for the other. This effect has attracted much attention in cavity optomechanics [1] and has been recently used to drive a slowly decaying microwave cavity mode into a coherent quantum state [2, 3, 4].

Our study refers to a pair of modes with frequency ratio close to 1:3. In many systems the coupling between such modes is comparatively strong; for symmetry reasons, in micro- and nano-mechanical systems it is often stronger than the coupling between the modes close to 1:2 resonance. We explore a system with driven nonlinear mode (mode 1) that is coupled to a linear mode (mode 2) with a frequency ratio $\omega_2 \approx 3\omega_1$. This model is amenable to analysis and reveals many generic features of IRs. The nonlinear mode is described by the simplest model that displays amplitude dependence of the vibration frequency, i.e., the Duffing oscillator, and the resonant coupling terms arise from a single-term potential $V_{int} = \alpha x_1^3 x_2$. Note that of all possible nonlinear coupling terms, V_{int} is sufficient to capture the essential features of the resonance, that is, it is the the key term in the normal form for this resonance [5, 6]. Both modes are taken to be lightly damped with exponential decay rates Γ_1 and Γ_2, respectively. The complex amplitude equations for this model are given by

$$\dot{A}_1 = -(\Gamma_1 + i\Delta\omega_1)A_1 + \frac{3i\gamma}{2\omega_F}|A_1|^2 A_1 + \frac{3i\alpha}{2\omega_F}A_2\bar{A}_1^2 - \frac{iF}{4\omega_F}, \tag{1}$$

$$\dot{A}_2 = -(\Gamma_2 + i\Delta\omega_2)A_2 + \frac{i\alpha}{6\omega_F}A_1^3, \tag{2}$$

where F and ω_F are the drive amplitude and frequency (from $(F\cos\omega_F t)$), and $\Delta\omega_1 = \omega_F - \omega_1,\ \Delta\omega_2 = 3\omega_F - \omega_2$ are the frequency detunings of modes 1 and 2 relative to the drive, respectively. Eqs. (1)-(2) can be considerably simplified for the case of $\Gamma_2/\Gamma_1 \gg 1$ (i.e., when mode 2 is relatively fast and acts as a thermal reservoir for mode 1). In the limit of $\Gamma_2/\Gamma_1 \gg 1$, we can disregard the time derivative of A_2 in Eq. (2) (since A_2 will have already reached the steady-state value on the time scale of the evolution of A_1 and will adiabatically follow A_1) and solve the resulting linear algebraic equation for A_2. Substitution of this A_2 into Eq. (1) yields the following reduced-order single-mode equation for A_1

$$\dot{A}_1 = -(\Gamma_1 + i\Delta\omega_1)A_1 + \frac{3i\gamma}{2\omega_F}|A_1|^2 A_1 - \frac{\alpha^2}{6\omega_F^2}(\Gamma_2 + i\Delta\omega_2)^{-1}|A_1|^4 A_1 - \frac{iF}{4\omega_F}. \tag{3}$$

Thus, we immediately see that the dynamics of mode 1 are augmented by a finite bandwidth ($-\Gamma_2 < \Delta\omega_2 < \Gamma_2$) nonlinear quintic damping given by $-\frac{\alpha^2}{6\omega_F^2}\frac{\Gamma_2|A_1|^5}{\Gamma_2^2+\Delta\omega_2^2}$. To gain insight into this nonlinear friction, which has linear and quintic, but not cubic, terms, we consider first the dynamics in the absence of driving, $F = 0$, for which we take $\omega_F \to \omega_1$ in the detuning parameters, and allow the complex amplitude to rotate in the complex plane with its Duffing amplitude-frequency dependency, $A_1 \to A_1 e^{i\Phi},\ \Phi = \frac{3\gamma}{2\omega_F}\int |A_1|^2 dt$. Of interest here is the manner in which mode 1 decays

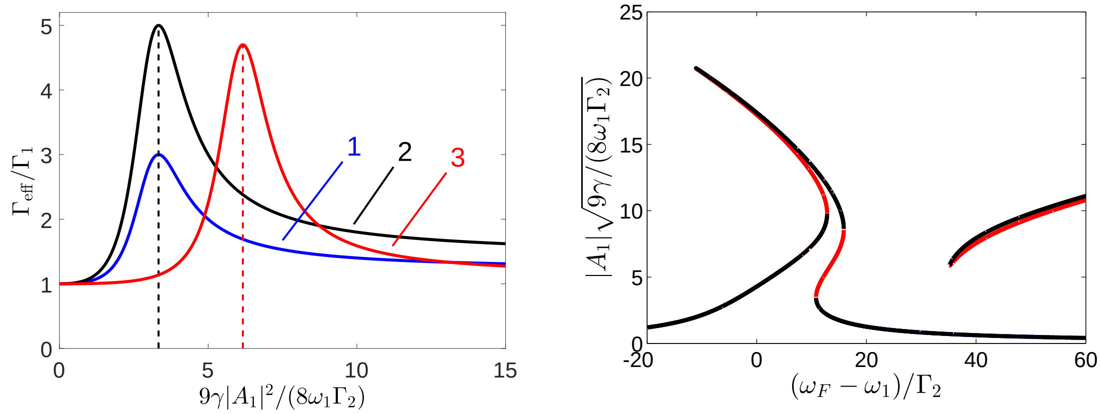

Figure 1: Left - dependence of the effective instantaneous decay rate $\Gamma_{\rm eff}$ on vibration amplitude $|A_1|$ of mode 1 in the adiabatic regime of fast decaying mode 2. Curves 1 and 2 refer to $\frac{3\omega_1-\omega_2}{\Gamma_2}=-3$ and $(\frac{\alpha}{9\gamma})^2\frac{\Gamma_2}{\Gamma_1}=0.2$ and 0.4, respectively. Curve 3 refers to $\frac{3\omega_1-\omega_2}{\Gamma_2}=-6$ and $(\frac{\alpha}{9\gamma})^2\frac{\Gamma_2}{\Gamma_1}=0.3$. Right - Mode 1 response curve for $\Gamma_2/\Gamma_1\gg 1$, stable/unstable steady-state responses denoted by black/red, curves, respectively. $\Gamma_2/\Gamma_1=50,\ \alpha/\gamma=1,\ (\omega_2-3\omega_1)/\Gamma_2=100$ and $F\sqrt{3\gamma/(32\omega_1^3\Gamma_2^3)}=55$.

from an initial amplitude in the presence of resonant nonlinear coupling. The left panel of Figure 1 shows the scaled instantaneous decay rate $\Gamma_{\rm eff}/\Gamma_1=\Gamma_1^{-1}d(\log|A_1|)/dt$ versus time for different parameter values. The figure shows two important features. First, it displays a peak value when the instantaneous frequency of mode 1 equals to $\omega_2/3$, that is, at the point of IR. The height of the peak increases with increasing coupling strength α and with increasing frequency detuning $|\Delta\omega_2|\propto\omega_2-3\omega_1$. Second, the instantaneous rate is nearly constant except near the IR condition. For small amplitudes linear damping holds, $\Gamma_{\rm eff}/\Gamma_1\approx 1$, while for large amplitudes the Duffing amplitude-frequency dependency leads to a non-resonant interaction. Yet, the decay rate, $\Gamma_{\rm eff}/\Gamma_1$, does not approaches unity. This is a consequence of the strongly nonlinear coupling, with the coupling energy $\propto|A_1|^3$.

Along with this anomalous dissipation, the backaction from a thermal reservoir leads to a higher-order nonlinear shift in the frequency, $\frac{\alpha^2}{6\omega_F^2}\frac{\Delta\omega_2|A_1|^4}{\Gamma_2^2+\Delta\omega_2^2}$, which has a strong effect on the response of the system to harmonic excitation. Note that for an upward sweep in the drive frequency, for a hardening mode 1, this frequency shift changes from softening at larger amplitudes ($\Delta\omega_2<0$) to hardening at lower amplitudes ($\Delta\omega_2>0$). This phenomenon leads to a peculiar response curve, a sample of which is shown in right panel of Figure 1. As can be seen from the figure, for sufficiently strong driving there is a region around $\omega_F\approx\omega_2/3$ where the response curve deviates sharply from the well-known single mode Duffing behaviour. This peculiarity is intimately related to the unforced amplitude-dependent frequency of mode 1 ($\omega_{1\rm eff}-\omega_1\propto|A_1|^2$), which, due to the coupling with mode 2, exhibits a "repulsion" of the vibration frequency away from the resonance where $3\omega_{1\rm eff}=\omega_2$. It also leads to an interesting hysteretic behavior. If we start from negative $\Delta\omega_1=\omega_F-\omega_1$ and increase ω_F, we move from the small amplitude branch until it ends and we jump to the high-amplitude branch. From there, if we continue to increase ω_F, we jump to the small-amplitude branch which goes to large positive $\Delta\omega_1$. The large amplitude branch for positive $\Delta\omega_1$ is isolated and cannot be accessed by varying ω_F at this level of drive.

As a final remark we note that Eq. (3) describes the dynamics of a single complex variable or equivalently, two real variables. It is seen from this equation that the stationary states of the system are either stable states or saddle points. Indeed, if we linearize Eq. (3) about a stationary state, we will see that the sum of the two eigenvalues of the corresponding characteristic equation is negative, which indicates that at least one of the eigenvalues that characterize the linearized motion has a negative real part. The system can switch between different stable states by varying parameters of the drive (or because of noise), but there are no states where the amplitude $|A_1|$ would oscillate. However, this result relies on the adiabatic approximation and applies only in the limit of a large ratio of decay rates $\Gamma_2/\Gamma_1\gg 1$.

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