Modes of vibration of nanobeams vibrating with large displacements and actuated by DC electric tensions

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Summary. The non-linear modes of vibration of electrostatically actuated nanobeams of rectangular cross section are investigated. The model employed is applicable to beams with relatively small length to thickness relations because it includes the transverse shear strain and the rotary inertia. Furthermore, non-local effects are considered in the structure and in the electrostatic force, in order to obtain a model that is applicable to beams with very small dimensions, and fringing field effects are included in the expression of the electrostatic force. It is shown that the non-local effects influence the degrees of hardening and softening. Modal interactions are found; the vibration frequencies and vibration amplitudes at which they start are altered by the non-local effects.

Introduction

Progresses on manufacturing methods turn the production of nanoelectromechanical systems (NEMS) increasingly feasible. Applications of electromechanical devices include force and mass sensing, switches, actuators and the detection of molecules or cells. Such applications often involve a vibratory behaviour of the system, therefore, it is important to analyse the modes of vibration of electromechanical devices. In the case of NEMS under electrostatic actuation, the modes of vibration are affected by two types of non-linearity: (1) the vibrating structure can oscillate with amplitudes that are large in comparison with its thickness, introducing geometrical non-linearity; (2) the electrostatic force is a non-linear function of the distance between the structure and the electrode.
NEMS contain structural elements with dimensions of the order of a few nanometres, where the classical elasticity theory is not necessarily applicable. Comparisons with molecular dynamics simulations, as the ones carried out in [1], indicate that the non-local elasticity theory of Eringen [2] leads to continuous models which are valid at small scales.
A Bernoulli-Euler, non-local, *p*-version finite element type model was proposed in [3] to analyse the non-linear modes of vibration of carbon nanotubes (CNTs) under electrostatic forces, with constant voltages. The former work is here extended by developing a non-local *p*-version FE based on Timoshenko's beam theory. The consideration of shear and of the inertia due to cross section rotation is important because it expands the domain of application of the *p*-version FE to shorter nanobeams. Furthermore, although the new *p*-version Timoshenko non-local FE also applies to CNTs, we focus here on vibrations of beams with rectangular cross section. In these beams, fringing field effects related to the electrostatic force appear [4] and their importance in the modes of vibration is here investigated.
By expanding the generalised coordinates in Fourier series and applying the harmonic balance method (HBM), the non-linear ordinary differential equations in the time domain are transformed to the frequency domain. With the help of a number of numerical tests, the influence of non-local effects, the importance of electrostatic forces (including, as said, the fringing fields), and the effect of the initial distance between the electrode and the nanobeam on the non-linear modes of vibration of nanobeams are discussed. Backbone curves, including secondary branches due to modal interactions, and shapes of vibration are investigated.

Equations of motion

In the non-local elasticity theory of Eringen, stresses at a point in an elastic continuum depend on the strains at all points of the domain. The general constitutive equations established by Eringen are integro–partial differential equations, but in the case of beams they can be written as differential equations [2]. If the transverse shear is considered, we have:

$$\sigma_{xx}(x,y,t) - \mu \frac{\partial^2 \sigma_{xx}(x,y,t)}{\partial x^2} = E\varepsilon_x(x,y,t) \tag{1}$$

$$\sigma_{xz}(x,y,t) - \mu \frac{\partial^2 \sigma_{xz}(x,y,t)}{\partial x^2} = G\gamma_{xz}(x,y,t) \tag{2}$$

where $\sigma_{xx}(x, y, t)$ is the axial stress and $\sigma_{xz}(x, y, t)$ the transverse shear stress; $\varepsilon_{xx}(x, y, t)$ and $\gamma_{xz}(x, y, t)$ are, respectively, the axial strain and the engineering transverse shear strain; E represents Young's modulus and G the transverse shear modulus. The non-local parameter is $\mu=(e_0a)^2$, where a represents an internal characteristic length (e.g., the lattice parameter) and e_0 is a constant that depends on the material.
The following expression gives the electrostatic force, $f_e(x,t)$, applied to beams of rectangular cross section [4]:

$$f_e(x,t) = -\frac{\varepsilon_0 b V^2}{2\left(d + w(x,t)\right)^2}\left(1 + 0.65\frac{d + w(x,t)}{b}\right), \tag{3}$$

it includes a first order fringing field effect correction. The meaning of the parameters and variables that appear in equa-

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tion (3) is: ε_0 - permittivity of free space; b – beam width; V- electric tension; d – undeformed gap space; $w(x, t)$ – transverse displacement.
The beam analogue of Von Kármán's strain displacement relations is employed and Hamilton's principle is applied, in conjunction with an expansion of displacement components in a p-version type approach [3]. The equations of motion obtained have the following form

$$\begin{bmatrix} \mathbf{M}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\phi \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_w(t) \\ \ddot{\mathbf{q}}_\phi(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{KL}_{w1} & \mathbf{KL}_{\phi 1} \\ \mathbf{KL}_{w2} & \mathbf{KL}_{\phi 2} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_w(t) \\ \mathbf{q}_\phi(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{KNL}(\mathbf{q}_w(t)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_w(t) \\ \mathbf{q}_\phi(t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{v}_{fe} \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad (4)$$

The mass matrices, **M**, and the non-linear stiffness matrix, **KNL**, contain terms due to the non-local effects. Both the linear, **KL**, and the non-linear stiffness matrices contain terms due to the electrostatic force. Vector $\mathbf{v}_{fe}$ is a constant vector due to the electrostatic force.

Illustrative results

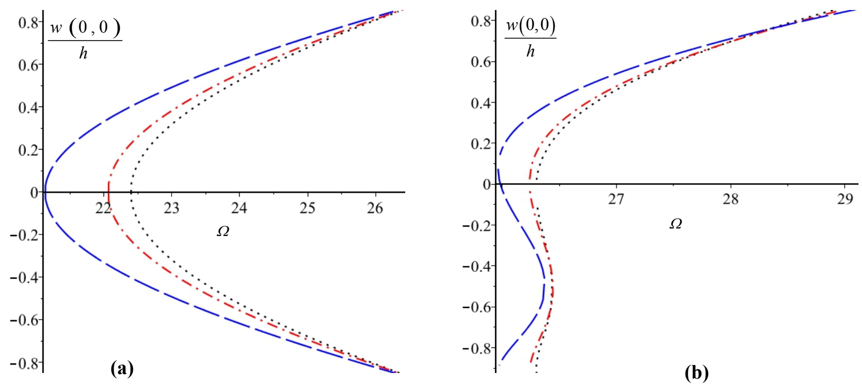

Figure 1: Non-dimensional transverse displacement $w(0,0)/h$, at beggining of a vibration cycle, versus non-linear natural frequency parameter Ω, for three values of the non-dimensional non-local parameter: ···· η=0, – · – η=0.2, — — η=0. 4. The values of the DC voltage are: **(a)** 0 V; **(b)** 10 V.

As an illustration of the many analysis that were carried out, Figure 1 shows the evolution of a non-linear natural frequency of vibration parameter, Ω [3], with the non-dimensional transverse displacement $w(0,0)/h$ (where h represents the thickness of the beam), with and without electrostatic forces. Letter η represents a non-dimensional non-local parameter ($\eta=e_0a/L$, with L representing the length of the beam). Due to the DC voltage, one finds softening which is followed by hardening. With and without DC voltage, the non-local effects increase the degrees of hardening and softening, because they alter the stiffness of the beam.

Concluding comments

A Timoshenko type, non-local, model for beams of rectangular cross section and under the action of electrostatic forces is derived. It is verified that the non-local effects alter the degrees of hardening and softening; the latter only occurs for high enough electric voltages. Non-local effects also influence the location of bifurcation points on the backbone curves. The influence of fringing fields, undeformed gap space and beam length on the non-linear dynamics of the non-local beams is analysed.

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References

[1] Hu Y.-G., Liew K.M., Wang Q. (2009) Nonlocal elastic beam models for flexural wave propagation in double-walled carbon nanotubes. *J. Appl. Phys.* **106**: 044301.
[2] Eringen A. (1983) On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *J. Applied Physics* **54**: 4703-4710.
[3] Ribeiro P. (2016) Non-local effects on the non-linear modes of vibration of carbon nanotubes under electrostatic actuation. Int. J. Non-Linear Mech **87**: 1-20.
[4] Osterberg P.M., Senturia S. D. (1997) M-TEST: A Test Chip for MEMS Material Property Measurement Using Electrostatically Actuated Test Structures. *J. Microelectromech. S.* **6**: 107-118.