Spectral Submanifolds and Exact Model Reduction for Nonlinear Beam Dynamics

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Summary. We use invariant manifold results on Banach spaces to conclude the existence of spectral submanifolds (SSMs) in a class of nonlinear, externally forced beam oscillations . Reduction of the governing PDE to the SSM provides an exact low-dimensional model which we compute explicitly. This model captures the correct asymptotics of the full, infinite-dimensional dynamics. Our approach is general enough to admit extensions to other types of continuum vibrations. The model-reduction procedure we employ also gives guidelines for a mathematically self-consistent modeling of damping in PDEs describing structural vibrations.

Extended Abstract

We will be concerned with the nonlinear partial differential equation

$$\begin{cases} u_{tt} - \mu u_{ttxx} = -\alpha u_{xxxx} + \beta u_{txx} - \gamma u - \delta u_t - f(u) - \varepsilon h(x,t) \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), \end{cases}$$

on the interval $(0, \pi)$, describing the vertical displacement u of a thin beam of length π supported on a nonlinear foundation with initial configuration u_0 and initial velocity distribution v_0.

The parameters relate to material constants of the beam, cf. [3]. The function h, which we assume to be ω-periodic in time, describes external forcing for some small parameter ε, while the numerical function $f : \mathbb{R} \to \mathbb{R}$ describes the nonlinear interaction of the beam with the foundation. An example would be a thin beam supported on a bed of cubic springs, cf. [4].

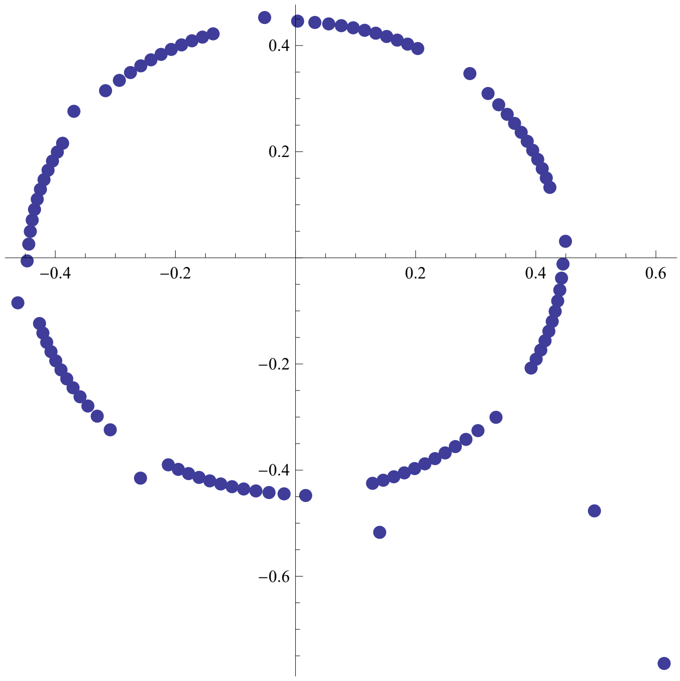

Figure 1: The spectrum of the linearized flow map

First, we show existence and uniqueness of solutions in a suitably chosen Sobolev space. This is done by classical semigroup techniques, making certain regularity assumptions on the nonlinearity. For $\varepsilon > 0$, existence of a periodic orbit to the full equation is proved by passing to a section of the return map and using a standard fixed-point argument. An energy estimate guarantees global existence of solutions, hence permitting us to define a flow map for this equation acting as a diffeomorphism on the underlying space.

Due to the nature of the damping, the spectrum of the linearization of the flow map is contained in unit circle, is bounded away from zero and has decreasing real parts, cf. Figure 1. This is only guaranteed by the inclusion of the parameter $\mu \neq 0$, accounting for small rotational inertia. Also, the balance of the elastic damping to the small rotational inertia is crucial to justify our calculations. In fact, for $\varepsilon = 0$, we may write down the spectrum of the linearized flow map explicitly:

$$\sigma(\mathcal{A}) = \left\{ \exp\left(-\frac{\beta n^2 + \delta}{2 + 2\mu n^2} \pm \sqrt{\left(\frac{\beta n^2 + \delta}{2 + 2\mu n^2}\right)^2 - \frac{\alpha n^4 + \gamma}{1 + \mu n^2}} \right) \right\}_{n \in \mathbb{N}} .$$

In the case of non-zero ε, analytical spectral perturbation theory justifies the assertion.

We observer that if the damping term becomes dominant, the equation loses its group property therefore invertability of the linearized flow map. This feature only becomes present in infinite-dimensional spaces, i.e. when studying partial differential equations, and is not a handicap in mechanical systems, i.e. ordinary differential equations.

In this setup, we may apply a theorem on the existence of submanifolds associated to spectral subspaces [1] to conclude that the dynamics may be described by the flow on some finite dimensional space. This is achieved by an asymptotic expansion of the invariant manifold and the projection onto a finite number of Fourier modes. In particular, we may conjugate the dynamics of the full partial differential equation to some system of ordinary differential equations. For physically reasonable parameter ranges, in particular assuming small damping, we infer that the invariant manifold indeed is a slow manifold for the full system. This procedure can be interpreted as an infinite-dimensional equivalent to [2].

We illustrate this technique on a concrete example, in which project on a single mode in the Fourier expansion, obtaining the dynamics of a weakly attracting focus, cf. Figure 2. Here, the expansion is carried out up to third order making use of the near-resonant nature of the eigenvalues of the linearization, as it is also done in [5] for finite-dimensional systems.

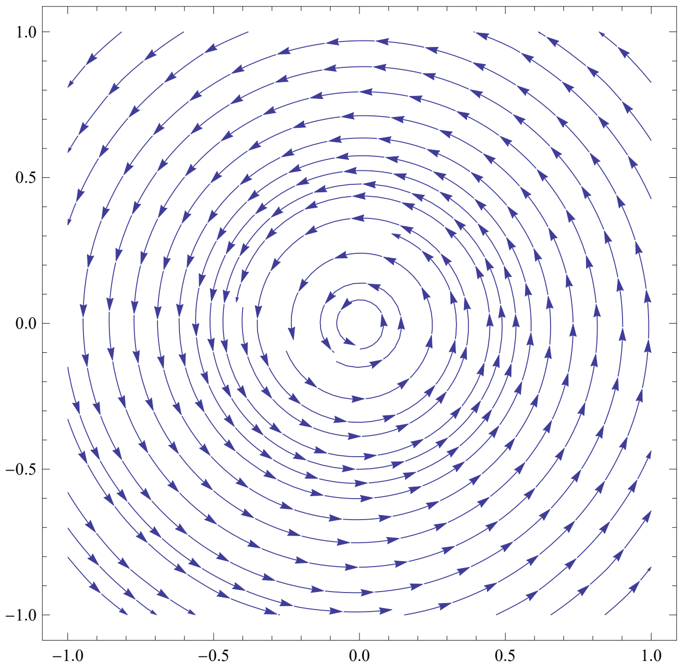

Figure 2: A typical phase portrait for the reduced system

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