Chain Fountain Dynamics

Friedrich Pfeiffer, Johannes Mayet
Applied Mechanics, Mechanical Engineering, TU-Muenchen, Germany

Summary. Chain fountains are known since long time, and many efforts have been taken to model and to explain the dynamics of such a chain fountain. A chain consists of many small elements starting from an inertial beaker, moving upwards by forming an arc and coming to an inertial position again after a rather long vertical distance. As the chain elements are all connected by a bearing type structure, they all have to go with the same velocity v. In the following we shall consider the stationary case and assume small chain elements (Fig. 1).

Modeling

As the problem is known since long time, famous colleagues have dealt with it, so for example Painleve [5] or Routh [7], who consider a continuous approach as many scientists today. Airy [1] solved a problem connected with underwater cables in the 19th century, where the cable connections between Europe and the US became important and feasible [8]. Some remarkable contributions come from Biggins [2], Calkin [3], Grewal[4] and Virga [9], to cite only a few.
This paper presents a multibody approach (MBS), and, by applying a limiting process also a belt approach. Geometry and forces are depicted in Figure 2, from which we derive the following equations of motion:

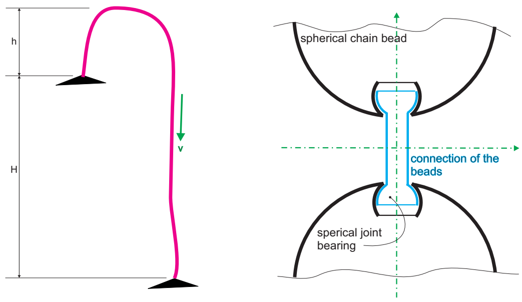

Figure 1: Chain Fountain and Chain Structure

$$\begin{aligned}\Delta m\ddot{x}_i =&F_i cos\beta_i - F_{i-1}cos\beta_{i-1} + F_{Ti}sin\beta_i - F_{T,i-1}sin\beta_{i-1}\\ \Delta m\ddot{y}_i =&F_i sin\beta_i - F_{i-1}sin\beta_{i-1} - F_{Ti}cos\beta_i + F_{T,i-1}cos\beta_{i-1}\\ &- \Delta mg. \end{aligned} \quad (1)$$

The frictional force writes $F_{Ti} = \mu(sign\dot{\alpha})F_i$, and the centrifugal effects with the force $F_{Ci} = \Delta m\frac{v^2}{R_i}$ and $\kappa_i = \frac{1}{R_i} = (\frac{d\alpha}{ds})_i$, are included in the acceleration terms. The initial conditions are $F_0 = k_0(\lambda v^2)$, $\alpha = \alpha_0, \beta = \beta_0$, The curvature κ_i depends on the change of the angle α with the trajectory coordinate s. The form of the chain fountain requires $\kappa_i < 0$. The factor k_0 takes impact losses during the pick-up process into account [9]. For the very small beads with their connection bars we discretize with ($ds \approx \Delta s \approx d_i,\ dx \approx d_i cos\beta_i,\ dy \approx d_i sin\beta_i,\ \kappa_i \approx (\frac{\Delta\alpha}{\Delta s})_i \approx (\frac{\Delta\alpha}{d})_i$). Introducing the accelerations $\ddot{x}_i = -v\dot{\alpha}_i sin\alpha_i,\ \ddot{y}_i = +v\dot{\alpha}_i cos\alpha_i$ and the abbreviations ($f_i = \frac{F_i}{\Delta mg},\ w_i = \frac{v^2}{gd_i}$), rearranging equations (1) and regarding the discretizations we come out with

$$\Delta\alpha_i \approx -\Big(\frac{cos\alpha_i + \mu_\alpha sin\alpha_i}{w_i - f_{i-1} - \frac{1}{2}sin\alpha_i + \frac{1}{2}\mu_\alpha cos\alpha_i}\Big). \quad (2)$$

Knowing $\Delta\alpha_i$ we find from the equations of motion also f_i. Moreover, within the framework of these approximations we can derive a force relation of the reduced form ($f_i \approx f_{i-1} + sin\beta_i$). Summing up this gives the well-known result ($F_E - F_0 \approx \lambda gH$). The difference of the two ground forces F_E and F_0 is the weight of the chain part below the arc.
We go now from the discretized solution with all details of the chain to a continuous solution with no details by applying the limiting process ($\Delta s_i \approx d_i \ \rightarrow\ ds$). This generates mathematically a continuous belt from the discrete chain with finite beads. The equations of motion (1) can then be put in a form writing

$$\alpha'(1-\bar{F}) = -(\frac{g}{v^2})cos\alpha, \qquad \bar{F}' = +(\frac{g}{v^2})sin\alpha, \qquad \text{with} \quad \bar{F} = \frac{F}{\lambda v^2}, \quad (.)' = \frac{d(.)}{ds}, \quad (3)$$

which have the solutions

$$\begin{aligned}\bar{F} = 1-(1-\bar{F}_0)(\frac{cos\alpha_0}{\cos\alpha}), \qquad tan\alpha =&\ \ tan\alpha_0 - qs \qquad \text{for} \quad (\alpha_0 \geq \alpha \geq 0), \ \text{left fountain arc},\\ tan\alpha =& -q(s-s_0) \qquad \text{for} \quad (0 \geq \alpha \geq \alpha_E), \ \text{right fountain arc}, \end{aligned} \quad (4)$$

which is another representation of the well-known catenary. Considering work, energy and power gives the same relation as just mentioned,namely ($F_E - F_0 \approx \lambda gH$). The centrifugal forces over the complete fountain have to carry the whole chain together with the forces at the ground. With Figure (2) and the results above we get the balance

$$\sum_{i=1}^{N} F_{Ci}cos\alpha_i = \sum_{i=1}^{N}\Delta mg - F_E sin\alpha_E + F_0 sin\alpha_0, \qquad s_E \approx (1-k_0)(sin\alpha_0 - sin\alpha_E)(\frac{v^2}{g}) + Hsin\alpha_E. \quad (5)$$

The above equations allow the solutions ($F_E = 0,\ \ \chi = \frac{gH}{v^2} \ = (\frac{cos\alpha_E}{cos\alpha_0}) - 1 < 0$)

$$k_0 = -\chi, \qquad (\frac{s_E g}{v^2}) = -\chi + 2(\frac{hg}{v^2}), \qquad (\frac{s_E}{H}) = -1 + (\frac{2}{\chi})(\frac{hg}{v^2}), \qquad (\frac{hg}{v^2}) = 1+\chi, \qquad (\frac{h}{H}) = (\frac{1+\chi}{\chi}). \quad (6)$$

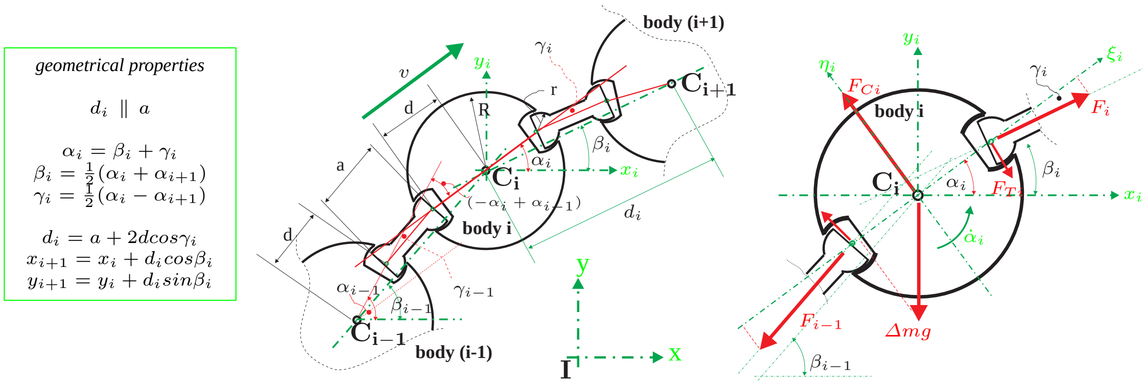

Figure 2: Chain Model - Geometry and Forces

Results and Conclusions

In addition to the calculations applying the dicretized and the continuous models some measurements have been performed. The comparisons are good and confirm theory.
In some parts of the literature we find for the ratio $(\frac{h}{H}) \approx -0.14$, which according to the above equations corresponds to $\chi \approx -0.875$. The magnitude χ depends, as all other parameters too, only on the angles at pick-up and put-down, $(\chi = \frac{gH}{v^2} = (\frac{\cos\alpha_E}{\cos\alpha_0}) - 1 < 0)$. Therefore a ratio $(\frac{h}{H}) \approx -0.14$ can only be realized by an angle ratio $(\frac{\cos\alpha_E}{\cos\alpha_0}) \approx 0.125$, which cannot be controlled precisely in experiments. Figure 3 firstly depicts a typical chain fountain with the relevant parameters, and it illustrates secondly these experiments and theories, where the ratio $(\frac{h}{H}) \approx -0.14$ is not given.

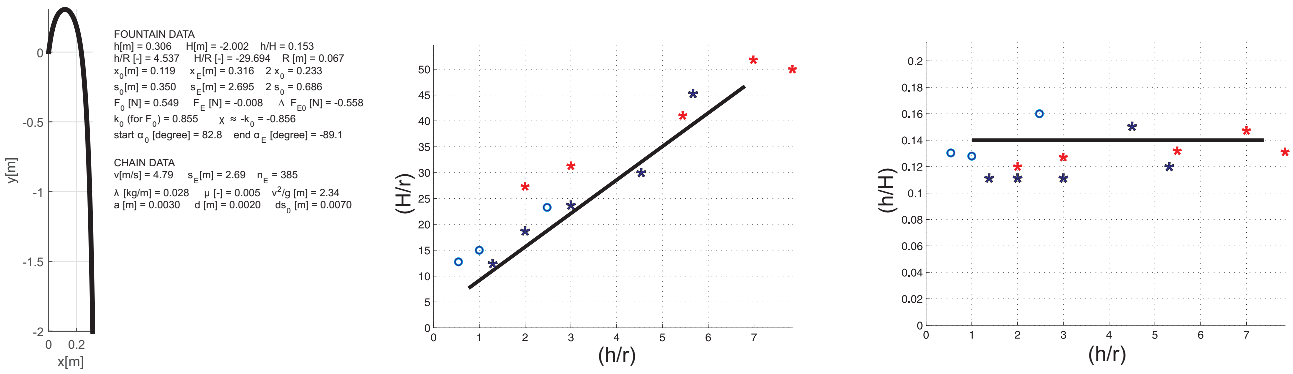

Figure 3: Results for the MBS and belt approximations compared with measurements: *left side* a typical chain fountain with the most important parameters; *right side* the chain fountain heights with black lines the $(\frac{h}{H} = \frac{1}{7})$-graphs, black stars MBS-approach, blue circles belt approximation, red stars measurements (scaling radius $r = (2r_{\alpha=0})$)

The dynamics of pick-up and put-down is the key to the motion of the chain. Looking at the different approaches, there are three facts without doubts: Firstly, at the pick-up and put-down points we have an impulsive character of the dynamical processes, which must be modeled accordingly. Secondly, a tension force starts at pick-up and disappears at put-down, acting on the whole fountain during motion. Thirdly, all approaches confirm the somehow surprising behavior of chain systems with variable masses including still some unsolved problems.
Considering only one chain element we have to accelerate this from v=0 to v=v coming out with an impulsive force $F_0 = k_0(\lambda v^2)$, which agrees with many results of the literature. The coefficient k_0 is an impact coefficient. With this approach realistic solutions are possible.

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