Dynamic analysis of a cantilever beam subject to a moving mass under unilateral constraint

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Summary. In this work we formulate a simple model for the description of the dynamical response of a vibrating beam under a moving mass with unilateral constraint. The effect of the impacts on the linear mode coefficients and on the overall nonlinear dynamics is illustrated by means of numerical simulations.

The Model

The analysis of the interaction between a cantilever beam and a moving mass, constrained to remain on the beam's surface, has been subject of numerous investigations in the literature [1, 2, 3, 4, 5, 6]. In particular, it is interesting to know how the traveling mass activates different linear modes of the beam, and how, roughly speaking, the mass indirectly couples these modes. In all previous investigations, the constraint is supposed bilateral, so that the mass is not allowed to leave the beam. A different situation arises when the moving mass is allowed to leave the beam as the force of contraint reverses its sign. The ball then freely moves in space under gravity, until it falls again on the beam, bounces back into space and so on. At this point, several behaviours are possible: the mass may undergo a sequence of bounces up to the end of the beam; it may bounce back and forth, due to the inflected profile of the beam at the bounce instants; it may enter a chattering situation [7, 8]. In order to analyze the dynamics described above, we consider a uniform beam of mass M and length L with clamped-clamped boundary conditions. The vertical profile of the beam $w(x,t)$ is governed by the beam equation; the mass which interacts with the beam is regarded as a point-mass m and its motion is governed by Newton's law. The coupled equations, in dimensionless form, which govern the system formed by the beam and the mass are then given by [6]

$$v^2\,\frac{\partial^2 w}{\partial x^2} + 2\,v\,\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 w}{\partial t^2} = \Phi - 1 \tag{1}$$

$$\frac{\partial^2 w}{\partial t^2} + \alpha^2\,\frac{\partial^4 w}{\partial x^4} = -\varepsilon\,\Phi\,\delta(x - v\,t) - 1 \tag{2}$$

where $x \in [0,1]$ is the space variable, $t \geq 0$ the time, v the horizontal velocity of the mass (initially supposed constant) and the parameters $\alpha^2 = E\,J/(M\,g\,L^2)$ and $\varepsilon = m/M$ have been introduced. The boundary conditions are $w(0) = w''(0) = w(1) = w''(1) = 0$. By eliminating the reaction Φ from equation (1), we finally obtain

$$\frac{\partial^2 w}{\partial t^2} + \alpha^2\,\frac{\partial^4 w}{\partial x^4} + \varepsilon\,\left(v^2\,\frac{\partial^2 w}{\partial x^2} + 2\,v\,\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 w}{\partial t^2}\right)\,\delta(x - v\,t) = -1 - \varepsilon\,\delta(x - v\,t). \tag{3}$$

Eigenfunction expansion

In order to solve Equation (3), we expand the profile function in the eigenfunctions of the free vibrating beam with the boundary conditions given above. Of course, the presence of the $\delta-$function in equation (3) makes this approach an approximation; a proper set of eigenfunctions could be, for example, those introduced in [9] and used also in [1]. However, we find that the error introduced by using the free vibrating beam eigenfunctions is small, and we leave the improvement on this point for future work. Before introducing the eigenfunction expansion, we split the profile function into a static term and an oscillating term, $w(x,t) = w_S(x) + \widetilde{w}(x,t)$, where the function w_S is the solution of the static equation and is given by $w_S(x) = -1/(24\,\alpha^2)\,(x^4 - 2\,x^3 + x)$. We finally obtain the governing equation for $\widetilde{w}$:

$$\frac{\partial^2 \widetilde{w}}{\partial t^2} + \alpha^2\,\frac{\partial^4 \widetilde{w}}{\partial x^4} + \varepsilon\,\left(v^2\,\frac{\partial^2 \widetilde{w}}{\partial x^2} + 2\,v\,\frac{\partial^2 \widetilde{w}}{\partial x \partial t} + \frac{\partial^2 \widetilde{w}}{\partial t^2}\right)\,\delta(x - v\,t) = -\varepsilon\,\left(1 + v^2\,w_S''\right)\,\delta(x - v\,t). \tag{4}$$

The beam eigenfunctions for the clamped-clamped boundary conditions are well known and are given by $\varphi_n(x) = \sqrt{2}\,\sin(\lambda_n x)$, with $\lambda_n = n\,\pi$, $n = 1, 2, ...$ By expanding the oscillating part of the profile function, $\widetilde{w}$, according to

$$\widetilde{w}(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\varphi_n(x) \tag{5}$$

we obtain, after some algebra,

$$\mathbf{A}(t)\cdot\ddot{\mathbf{c}} + 2\,v\,\varepsilon\,\mathbf{B}(t)\cdot\dot{\boldsymbol{c}} + \mathbf{D}(t)\cdot\boldsymbol{c} = -\varepsilon\,\mathbf{f}(t) \tag{6}$$

where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$ and the vector $\boldsymbol{f}$ are given by

$$A_{mn} = \delta_{mn} + \varepsilon\,\varphi_m(v\,t)\,\varphi_n(v\,t) \qquad B_{mn} = \varphi_m(v\,t)\,\varphi_n'(v\,t)$$

$$D_{mn} = \alpha^2\,\lambda_m^4\,\delta_{mn} - \varepsilon v^2\,\lambda_n^2\,\varphi_m(v\,t)\,\varphi_n(v\,t) \qquad f_m = \left[1 + v^2 w_S''(v\,t)\right]\,\varphi_m(v\,t),$$

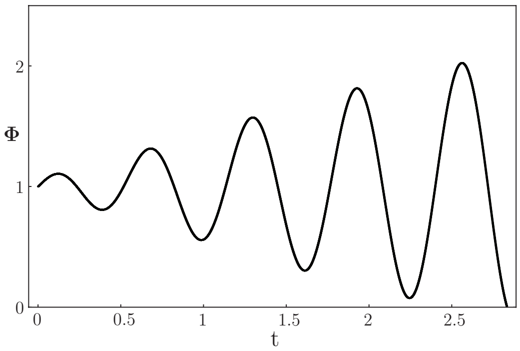

Figure 1

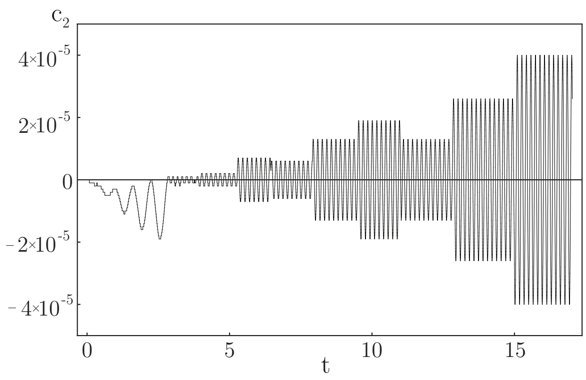

Figure 2

$n, m = 1, 2, ..., N$. Equations (6) are solved by a Runge-Kutta technique in time, until the force of reaction Φ remains positive. Figure 1 portrays the behaviour of the force of contraint as function of time, as given by our numerical simulations with $\alpha = 1$, $\varepsilon = 0.01$ and $v = 0.1$, with the beam initially vibrating in the fundamental mode, from $t = 0$ until it changes sign. When Φ becomes negative, the moving mass leaves the beam and subsequently undergoes a sequence of bounces whose outcome depends on the ball's velocity and on the beam's profile at the time of the successive impacts.

The impacts

Once the ball leaves the surface of the beam, the beam itself vibrates according to the free beam equation and the ball undergoes a sequence of impacts until it exits the beam; the coefficients $c_n(t)$ which appear in equation (5) for the beam, are given by

$$c_n(t) = A_n \cos\omega_n(t - t_k) + B_n \sin\omega_n(t - t_k) \tag{7}$$

where $\omega_m = \alpha\,\lambda_n^2$ and A_n and B_n are determined by the values of $c_n(t)$ and $\dot{c}_n(t)$ immediately after the previous impact. Then, we need to find the relationship between the velocities of the ball before and after the bounce and the relationship between the expansion coefficients of the beam before and after each impact. By assuming that the tangential (with respect to the beam profile) component of the ball's velocity is conserved, while the normal component reverses its sign and attenuates its modulus by a restitution coefficient $r < 1$ we find that the coefficients $c_n(t)$ and $\dot{c}_n(t)$ are given by

$$c_m(0+) = c_m(0-) \tag{8}$$

$$\dot{c}_m(0+) = \dot{c}_m(0-) + \frac{\kappa_m}{\omega_m} \tag{9}$$

where $\kappa_m = \varepsilon\,(1 + r)\,v_\perp\,\varphi_m(x_k)$, with $v_\perp$ the perpendicular velocity of the mass at the impact. Here $c_m(0+)$ and $\dot{c}_m(0+)$ denote the coefficients and their derivatives after the impact, while $c_m(0-)$ and $\dot{c}_m(0-)$ denote their values before the impact. Equations (8) and (9) provide the initial values of the expansion coefficients for the free vibrations of the beam between each impact and the next one. The calculation of the ball's velocities after each impact is rather standard and we do not report it here. Figure 2 shows the result of the numerical simulation for the second vibration mode of the beam; the figure clearly displays the initial trait, where the ball is attached to the beam, and the effect of all subsequent impacts of the mass with the beam, resulting in the time derivatives of the coefficients being discontinuous.

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