Lyapunov stability and existence results of measure differential inclusions - applications in nonsmooth mechanics with singular mass matrices

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Summary. Introduced by Moreau [5] the nonsmooth and setvalued framework of measure differential inclusions allows to describe mechanical systems including friction and impacts. Existence results for the equations of motion of nonsmooth mechanics based on this modelling approach always assume the regularity of the mass matrix and the independence of all constraints. Studies on Lyapunov stability of this systems [4] postulate the same requirements. But in our center of interests are special applications with singular mass matrices or redundant constraints including examples of [6]. For analytical investigations on this certain problems it is necessary to generalize the present theory for existence and stability to the linear implicit formulation of measure differential inclusions like it was already done for systems with bilateral constraints [3] and unilateral constraints with a finite number of impacts [1].

Mechanical systems with unilateral constraints and impact dealing with singular mass matrices

Following Glocker [2], mechanical systems with unilateral constraints and impacts can be described by a measure differential inclusion

$$dq = v\,dt, \tag{1a}$$
$$M(q)dv = f(q,v)dt + G^T(q)di, \tag{1b}$$
$$-di \in N_{C_N}(\xi)dt + N_{C_N}(\xi)dr \tag{1c}$$
$$\xi = G(q)(v^+ + \epsilon v^-), \epsilon \in [0,1] \tag{1d}$$

with respect to the absolute continuous position $q : [0,T] \to \mathbb{R}^n$ and the velocity $v : [0,T] \to \mathbb{R}^n$ of locally bounded variation with a Lebesgue decomposition for the differential measure $dv = \dot{v}dt + (v^+ - v^-)dr$ refering to the Lebesgue measure dt and the jump measure $dr := \sum_j \delta_{t_j}$ containing all Dirac measures δ_{t_j} of the discontinuity points of v. The set $N_{C_N}(x)$ defines the normal cone of all non-negative real vectors standing orthogonal to x, e.g. $\{y \in \mathbb{R}^n : y^T(x^* - x) \leq 0, \forall\, x^* \geq 0\}$.

Within many publications dealing with mechanical systems the mass matrix $M(q)$ is assumed to be positive definite, hence non-singular. But the number of articles considering applications with singular mass matrices is steadily increasing [6]. An existence result for positive semidefinite $M(q)$ with the additional assumption

$$\ker M(q) \cap \ker G(q) = \{0\} \tag{2}$$

can be found in [1]. The physical interpretation behind this formula is that any possible movement is associated with a positive kinetic energy. With the same condition (2) we give a generalized existence and uniqueness result for measure differential inclusion to involve also the case of an infinite number of discontiuity points in advance to [1]. Singular mass matrices often arise while the modelling process is simplified and structured by using non minimal coordinates. One application is a two degree-of-freedom system of two masses connected by springs [6]. To set up the equations of motion, it proves to be advantageous to consider two separated subsystems and connect them by a constraint $q_1 = x_1 + d$.

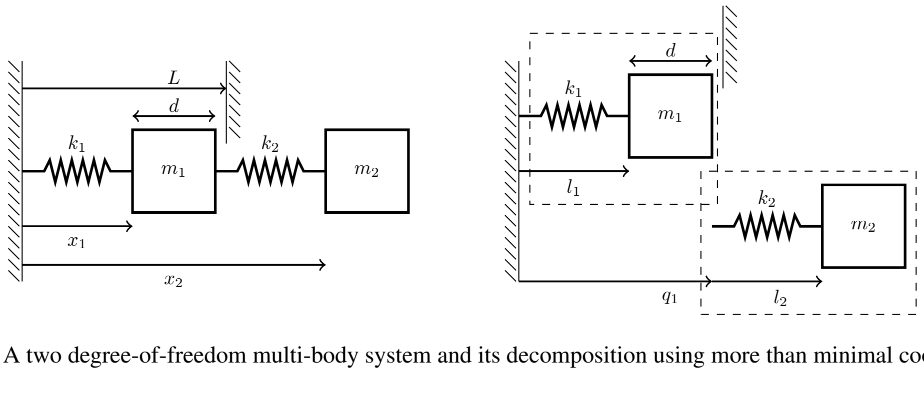

Figure 1: A two degree-of-freedom multi-body system and its decomposition using more than minimal cooridinates

With the notations of Figure 1 and $\bar{x}_1 = x_1 - l_1, \bar{q}_2 = q_2 - l_2$ the equations of motion are obtained by

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{\bar{x}}_1 \\ \ddot{q}_1 \\ \ddot{\bar{q}}_2 \end{pmatrix} = \begin{pmatrix} -k_1\bar{x}_1 - \lambda_1 - \lambda_2 \\ \lambda_1 \\ -k_2\bar{q}_2 \end{pmatrix}, \qquad \begin{array}{c} q_1 - \bar{x}_1 - l_1 - d = 0, \\ \\ 0 \leq \lambda_2 \perp L - \bar{x}_1 - l_1 - d \geq 0. \end{array} \tag{3}$$

The advantage of this modelling strategy gets stronger with larger problem dimension. Based on our analytical results this problem has a unique solution if (2) holds. This fact motivates the following stability analysis of equilibriums of mechanical systems with singular mass matrices.

Lyapunov stability of linear implicit measure differential inclusions

A generalized problem to (1a)-(1d) are the linear implicit measure differential inclusions

$$A(x)dx \in \Gamma(x) \tag{4}$$

with a measure- and multi-valued right hand side and an admissible set $Z = \{x : \Gamma(x) \neq 0\}$. The challenge of stability analysis of such systems is their impulsive character and the resulting state discontinuties. To pick up these properties Leine, van de Wouw [4] define a equilibrium x^* of (4) as stable if for every $\epsilon > 0$ there exists a $\delta > 0$ with for all starting values $x(t_0) = x_0 \in Z$:

$$\|x_0 - x^*\| < \delta \Rightarrow \|x(t) - x^*\| < \epsilon, \quad \text{for almost all } t \geq t_0$$

and all solutions $x(t)$ of (4) to x_0. For all discontinuity points of the function x the conclusion does not need to be true. The results of [4] are restructured to apply them to the implicit formulation (4).

Theorem 1: *An equilibrium x^* of (4) is stable in the sense of Lyapunov if there exists a lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and a neighborhood $U := B(x^*, h) \cap Z$ for a $h > 0$ with (i) $V(x) \geq 0, \forall\, x \in U, V(x^*) = 0$, (ii) $dV(x) \leq 0, \forall\, x \in U$, (iii) the set $\{x \in U : V(x) = 0\}$ contains only x^*.*

In [4] instead of (i) even positive definiteness of V is required such that (iii) becomes no longer necessary. But this approach is not applicable for systems with singular $A(x)$. The contradiction proof is based on the positively invariant property of sublevel sets of functions with bounded variation. Furthermore, Theorem 1 can be generalized to attractivity by replacing (iii) by the condition that $\{x \in U : dV(x) = 0\}$ contains only x^*.

Numerical test results

As in [4], the extended total mechanical energy

$$V(q, v) := 0.5v^T M(q)v + U(q) + \psi_Z(q) \tag{5}$$

with the positive definite $U(q)$ with respect to q^* and the indicator function ψ of the admissible set Z is chosen as a Lyapunov candidate. It can be shown that (5) fulfills the assumptions of Theorem 1 if $M(q)$ is for all q positive semidefinite and in addition (2) is satisfied. Considering the application in Figure 1 with $L = l_1 + d$ the system has an equilibrium point in $q^* := (\bar{x}_1, q_1, \bar{x}_2) = (0, L, 0), \dot{q}^* = (0, 0, 0)$. This is stable and attractiv, i.e. every solution to any (q_0, v_0) converges to (q^*, v^*) for $t \to \infty$, like it is underlined by Figure 2 showing $\bar{x}_1, \dot{\bar{x}}_1$ with different initial values.

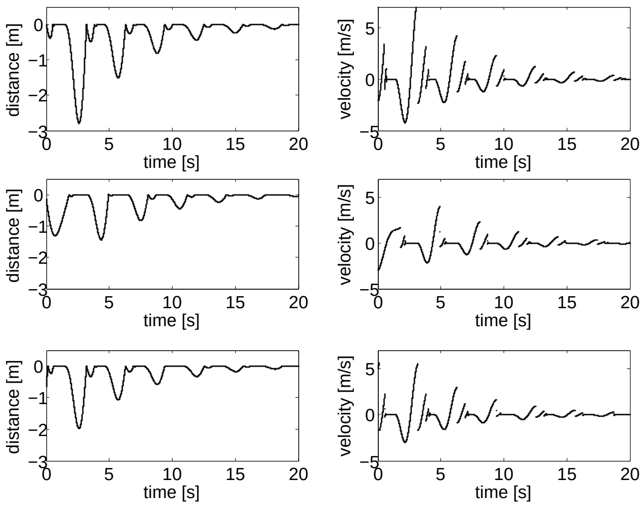

Figure 2: Solution components $\bar{x}_1, \dot{\bar{x}}_1$ of (3) for three different initial values underlining attractivity

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