Experimental Tracking of Limit-point Bifurcations using Control-based Continuation

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Summary. Control-based continuation (CBC) is a means of applying numerical continuation directly to a physical experiment allowing bifurcation analysis of the physical structure without the need for a mathematical model. In this paper, CBC is exploited to directly locate limit-point bifurcations of a periodically forced oscillator and track them as forcing parameters are varied. The proposed method is demonstrated on a single-degree-of-freedom mechanical system with a softening-hardening nonlinear stiffness characteristic.

Introduction

Experimental testing is an integral part of the development cycle of most engineering structures and is sometimes a mandatory step for certification (see, for example, aircraft ground vibration testing). Structures are not only tested to make sure they can withstand the external loads they will endure when in operation, but also to extract specific dynamical characteristics key to the development and validation of the mathematical models used to optimise their design.

Bifurcations represent stability boundaries where dramatic qualitative and quantitative changes in the dynamics of a system can occur and, as such, they are often key to the understanding of a system's dynamics. For instance, limit-point (LP) bifurcations were used to predict the existence of isolated periodic solutions in the nonlinear frequency response of several systems (see, for instance, [1, 2]). LP bifurcations also mark out the region where hysteretic behaviour can be observed when sweeping back and forth the resonance of nonlinear mechanical systems.

This paper proposes a simple method to directly track LP bifurcations in the experiment using control-based continuation (CBC). A single-degree-of-freedom nonlinear system with an adjustable softening-hardening restoring force is considered for demonstration. It is envisioned that the great deal of information carried by LP bifurcation curves can then be useful to estimate and update model parameters.

Tracking limit-point bifurcations using control-based continuation

Without the need for a mathematical model, control-based continuation (CBC) is a way of applying the concepts behind numerical continuation to a physical system [3]. At a basic level, numerical continuation tracks the solutions of a *zero problem* $f(\mathrm{x}, \lambda) = 0$ where $\mathrm{x} \in \mathbb{R}^c$ are the system states and $\lambda \in \mathbb{R}^d$ are the system parameters. To apply a similar idea to an experiment, there are two key challenges to overcome. (1) In general, it is not normally possible to set all the states x of the physical system and so it is not possible to evaluate f at arbitrary points. (2) The physical system must remain around a stable operating point while the experiment is running.

To address these challenges, a feedback controller is used to stabilise the system and the control target (or reference signal) acts as a proxy for the system state. The feedback control signal takes the general form $u(t) = g(x^*(t) - x(t))$ where $x^*(t)$ is the control target, $x(t)$ is the measured (or estimated) state for x and g is a suitable control law. The challenge here is to make the controller noninvasive such that the position in parameter space of any invariant sets such as equilibria and periodic orbits is not affected by the controller and is identical to the uncontrolled system of interest. This requirement for non-invasiveness defines the zero problem used in the experiment; a control target must be found such that the control action $u(t) \equiv 0$.

In this way, CBC can measure the periodic orbits of the tested specimen, regardless of their stability in the original uncontrolled system. These periodic orbits can then be followed in parameter space using path-following techniques. To track LP bifurcations, the presence of noise will, in general, prevent any attempt to directly measure the bifurcation point. To circumvent this issue, the proposed method relies on the geometric nature of LP bifurcations (extremum in the bifurcation parameter) and collects suitably positioned data points (i.e. periodic orbits) to estimate the actual position of the bifurcation using a polynomial regression [4]. More precisely, the proposed method works as follows: **Step I.** n data points equally distributed around a predicted bifurcation point are measured. **Step II.** A polynomial regression is built based on the n data points collected in Step I and the position of the LP bifurcation point is then estimated as one of the roots of the polynomial's first-order derivative. In practice, a cubic polynomial is chosen, which has the advantage of allowing the detection of two LP bifurcations within the data, as it can be the case when another LP bifurcation curve is crossed (for instance, close to a codimension-two cusp bifurcation). **Step III.** The need for additional data is assessed based on the identified LP and the quality of the cubic fit (using cross validation techniques). **Step IV.** The position of a new LP bifurcation point is predicted using the position of the previous two bifurcation points and linear extrapolation.

Experimental demonstration

The method is demonstrated on a single-degree-of-freedom (SDOF) oscillator made of a thin steel plate clamped to an aluminum armature at one end. At the other end of the plate, two sets of neodymium magnets are attached. The moving magnets interact with an iron laminated stator and a coil, introducing a complex nonlinear restoring force with softening-hardening characteristic. The nonlinearity can be adjusted by changing the air gap between the magnets and the iron stator such that the relative importance of the softening region over the hardening region increases for smaller gaps. The system is excited at its base using an electrodynamic shaker.

Figure 1 presents the LP bifurcation curves measured experimentally when the air gap gives a hardening nonlinearity. Oscillations were controlled using a proportional and derivative displacement feedback law. The results obtained with the proposed method (coloured curves) are compared to reference LP curves calculated from detailed data sets capturing the complete response surface and Gaussian process (GP) regression [5]. There is a very good agreement between the directly measured and GP calculated bifurcation curves.

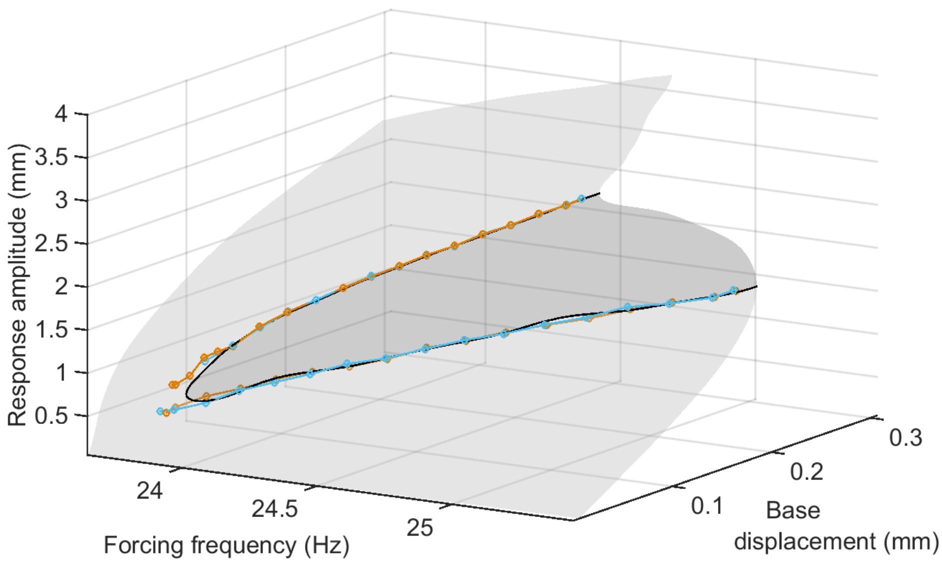

Figure 1: 3D projection of the fold bifurcation curve obtained using Gaussian process regression (—). Curves measured with the proposed method and $n = 5$ (—), $n = 7$ (—) data points.

Conclusions

In this paper, we propose a simple method based on control-based continuation to track limit-point bifurcation curves directly during experimental tests. The method is demonstrated on a single-degree-of-freedom oscillator for several configurations of the nonlinearity. The results is shown to agree very well with reference bifurcation curves calculated from detailed data sets capturing the complete response surface and Gaussian process regression. Compared to this latter approach, the proposed method is shown to considerably reduce the overall testing time. Moreover, it is shown that the new direct limit-point bifurcation tracking method avoids the artificial distortions observed in the limit-point curves calculated from the Gaussian process regression. In comparison to other ways of obtaining these results, the paper also shows that CBC is highly versatile and reliable. In contrast, obtaining limit-point bifurcation curves without the use of a closed-loop controller is tedious and error-prone, due to the fact that the curve itself is a stability boundary that can only be approached from one direction.

References

[1] Kuether, R. J., et al. (2015). Nonlinear normal modes, modal interactions and isolated resonance curves. *J. of Sound and Vibration* **351**: 299-310.

[2] Gatti, G. (2016). Uncovering inner detached resonance curves in coupled oscillators with nonlinearity. *J. of Sound and Vibration* **372**: 239-254.

[3] Sieber J., Krauskopf, B. (2008). Control based bifurcation analysis for experiments. *Nonlinear Dynamics* **51(3)**: 365-377.

[4] Renson L., Barton D. A. W., Neild S. A. (2017). Experimental tracking of limit-point bifurcations and backbone curves using control-based continuation. *International Journal of Bifurcation and Chaos*, in press.

[5] Renson L., et al. (2016). Robust identification of backbone curves using control-based continuation. *J. of Sound and Vibration* **347**: 145-158.