Stability of fractional positive continuous-time and discrete-time nonlinear systems

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Summary. The stability of fractional positive continuous-time and discrete-time nonlinear systems is addresed. The sufficient conditions for asymptotic stability are established by the use of an extension of the Lyapunov method to fractional positive nonlinear systems. The stability criteria are demonstrated on numerical examples.

Introduction

A dynamical system is called fractional if it is described by fractional order differential equation [13, 21-24]. The fundamentals of fractional differential equations and systems have been given in [21-24]. The stability of fractional linear systems have been analyzed in [2-4, 6, 7, 13, 20, 27, 28].
In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 5, 9].
Positive linear systems with different fractional orders have been addressed in [10, 16, 26]. Stability of fractional positive linear systems has been investigated in [1, 5, 8]. Descriptor (singular) fractional linear systems have been analyzed in [12, 17, 18, 25]. The stability of a class of nonlinear fractional-order systems has been analyzed in [6, 14, 29]. Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems has been presented in [11].
In this paper the stability of fractional positive continuous-time and discrete-time nonlinear systems will be addressed. The paper is organized as follows. In section 2 the stability of fractional positive continues-time nonlinear continuous-time systems is analyzed and in section 3 the stability of discrete-time nonlinear systems. Concluding remarks are given in section 4.
The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, Z_+ - the set of nonnegative integers, M_n - the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Stability of fractional positive continuous-time nonlinear systems

Consider the fractional nonlinear continuous-time system

$$ {}_0D_t^\alpha x(t)=\frac{d^\alpha x(t)}{dt^\alpha}=Ax(t)+f(x(t)),\quad 0<\alpha<1 \tag{1}$$

where

$$ {}_0D_t^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau,\ \ \dot{x}(\tau)=\frac{dx(\tau)}{d\tau} \tag{2}$$

is the Caputo fractional derivative of the order α of the state vector $x(t)\in\Re^n$ and

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ \mathrm{Re}(z)>0 \tag{3}$$

is the Euler gamma function, $A\in\Re^{n\times n}$ and $f(x(t))\in\Re^n$ is the continuous vector function of $x(t)$.
It is assumed that the nonlinear equation (1) has a solution $x(t)$ and the Lipschitz condition

$$\begin{gathered}|f_i(t,x_1',...,x_n')-f_i(t,x_1'',...,x_n'')|<c\{|x_1'-x_1''|+...+|x_n'-x_n''|\}\geq 0,\\ t\geq 0,\quad i=1,2,...,n\end{gathered} \tag{4}$$

is satisfied for some constant c and x_k', x_k'' for $k = 1,2,...,n$ are some values of the components of the state vector $x_k(t)$.

Definition 1. The fractional nonlinear system (1) is called (internally) positive if $x(t) \in \Re_+^n$ for all initial conditions $x_0 = x(0) \in \Re_+^n$.

Theorem 1. The fractional nonlinear system (1) is positive if and only if

$$A \in M_n \text{ and } f(x(t)) \in \Re_+^n \text{ for } x(t) \in \Re_+^n. \tag{5}$$

Proof. It is well-known [13] that if $f(x(t)) = 0$ then $x(t) \in \Re_+^n$, $t \geq 0$ if and only if $A \in M_n$ and $x_0 \in \Re_+^n$. By assumption the equation (1) has a solution and the condition (4) is satisfied. Using the Picard method it can be shown that the equation has a solution $x(t) \in \Re_+^n$ if the condition (5) is met. □
Consider the positive continuous-time nonlinear system

$$\frac{d^\alpha}{dt^\alpha} x = Ax + f(x), \; 0 < \alpha < 1 \tag{6}$$

where $x = x(t) \in \Re^n$, $A \in M_n$, $f(x) \in \Re_+^n$ is a continuous and bounded vector function and $f(0) = 0$.
Definition 2. The positive fractional continuous-time nonlinear system (6) is called asymptotically stable in the region $D \in \Re_+^n$ if $x(t) \in \Re_+^n$, $t \geq 0$ and

$$\lim_{t \to \infty} x(t) = 0 \text{ for any finite } x_0 \in D \in \Re_+^n. \tag{7}$$

To test the asymptotic stability of the positive system (6) the extension of the Lyapunov method will be used. As a candidate of Lyapunov function we choose

$$V(x) = c^T x > 0 \text{ for } x = x(t) \in \Re_+^n, \; t \geq 0 \tag{8}$$

where $c \in \Re_+^n$ is a vector with strictly positive components $c_k > 0$ for $k = 1,...,n$.
Using (8) and (6) we obtain

$$\frac{d^\alpha}{dt^\alpha} V(x) = c^T \frac{d^\alpha}{dt^\alpha} x = c^T [Ax + f(x)] < 0 \tag{9}$$

for

$$Ax + f(x) < 0 \text{ for } x \in D \in \Re_+^n, \; t \geq 0 \tag{10}$$

since $c \in \Re_+^n$ is strictly positive vector.
Therefore, the following theorem has been proved.
Theorem 2. The positive continuous-time nonlinear system (6) is asymptotically stable in the region $D \in \Re_+^n$ if the condition (10) is satisfied.
Example 1. Consider the nonlinear system (6) with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad f(x) = \begin{bmatrix} x_1 x_2 \\ x_2^2 \end{bmatrix}. \tag{11}$$

The nonlinear system (6) with (11) is positive since $A \in M_2$ and $f(x) \in \Re_+^2$ for all $x \in \Re_+^2$, $t \geq 0$.
In this case the condition (9) is satisfied in the region D defined by

$$D := \{x_1, x_2\} = \begin{bmatrix} -2x_1 + x_2 + x_1 x_2 \\ x_1 - 3x_2 + x_2^2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

From (12) we have

$$x_1(2-x_2) > x_2 > 0 \text{ and } 0 \le x_1 < (3-x_2)x_2 . \tag{13}$$

The region D is shown on the Fig. 1.

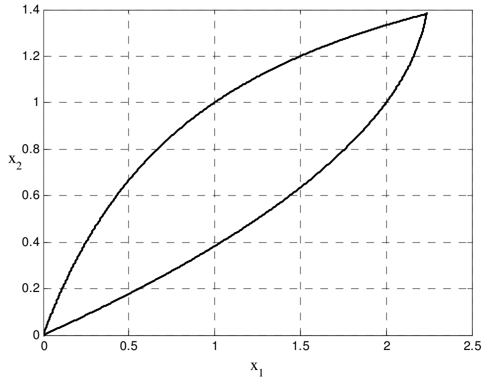

Fig. 1. Stability region (inside the curved line).

By Theorem 2 the positive fractional nonlinear system (6) with (11) is asymptotically stable in the region (12).

Stability of fractional positive discrete-time nonlinear systems

Consider the fractional discrete-time nonlinear system

$$\Delta^\alpha x_i = Ax_i + f(x_{i-1}, u_i) , \ 0 < \alpha \le 1 , \ i \in Z_+ = \{0,1,..\} , \tag{14a}$$

$$y_i = g(x_i, u_i) , \tag{14b}$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i} a_j^\alpha x_{i-j} , \tag{14c}$$

$$a_j^\alpha = (-1)^j \binom{\alpha}{j} = (-1)^j \begin{cases} 1 & \text{for} \quad j = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,... \end{cases} \tag{14d}$$

is the α -order difference of x_i, $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors, $A \in \Re^{n\times n}$ and $f(x_{i-1}, u_i) \in \Re^n$, $g(x_i, u_i) \in \Re^p$ are vector functions continuous in x_i and u_i.
Note that the fractional difference (14c) is defined in the point "i" not as usually in the point "i + 1" [13, 22].
Substituting (14c) into (14a) we obtain

$$\sum_{j=0}^{i} a_j^\alpha x_{i-j} = Ax_i + f(x_{i-1}, u_i) \tag{15a}$$

and

$$x_i = \sum_{j=1}^{i} A_1 c_j^\alpha x_{i-j} + f_1(x_{i-1}, u_i) , \ i \in Z_+ , \tag{15b}$$

where

$$\begin{aligned} &c_j^\alpha = -a_j^\alpha , \ j = 1,...,i, \\ &A_1 = [I_n - A]^{-1} \in \Re^{n\times n}, \ f_1(x_{i-1}, u_i) = A_1 f(x_{i-1}, u_i). \end{aligned} \tag{15c}$$

Assuming $x_i = 0$, $i = 1,2,...$ from (15b) for $i = 0$ we have

$$x_0 = f_1(0, u_0) . \tag{16}$$

Therefore, the initial condition x_0 is related with u_0 by (16).

Lemma 1. The matrix

$$A_1 = [I_n - A]^{-1} \in \Re_+^{n\times n} \tag{17}$$

if and only if the positive linear system

$$x_{i+1} = Ax_i \,,\; A \in \Re_+^{n\times n} \tag{18}$$

is asymptotically stable.

Proof. The positive discrete-time linear system (18) is asymptotically stable if and only if the matrix $A - I_n \in M_n$ is asymptotically stable (is Hurwitz) [6]. The condition (17) is satisfied if the system (18) is asymptotically stable [6]. □

Theorem 3. The solution x_i of the equation (15b) for given initial condition $x_0 \in \Re^n$ and input $u_i \in \Re^m$, $i \in Z_+$ has the form

$$x_i = \Phi_i x_0 + \sum_{j=1}^{i} \Phi_{i-j} f_1(x_{j-1}, u_j)\,, \tag{19a}$$

where

$$\Phi_j = \sum_{k=1}^{j} c_k^{\alpha} A_1 \Phi_{j-k}\,,\; j = 1,2,...,i\,,\; \Phi_0 = I_n\,. \tag{19b}$$

Proof. The proof can be accomplished by induction or by checking that (19) satisfies the equation (15b). □

In particular case for linear system

$$x_i = \sum_{j=1}^{i} A_1 c_j^{\alpha} x_{i-j} + B_1 u_i\,,\; i \in Z_+\,,\; B_1 \in \Re^{n\times m} \tag{20}$$

the solution x_i has the form

$$x_i = \Phi_i x_0 + \sum_{j=1}^{i} \Phi_{i-j} B_1 u_j \tag{21}$$

and the matrix Φ_j is given by (19b).

Remark 1. The solution x_i of the equation (15b) can be computed using the formulae (19) iteratively for $i = 1,2,...$ and substituting x_{j-1} given by (19a) into the vector function $f_1(x_{j-1}, u_j)$ for $i = 1,2,...$.

Definition 3. The discrete-time nonlinear system (14) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 4. The discrete-time nonlinear system (14) is positive if and only if $0 < \alpha \le 1$, the matrix $A \in \Re_+^{n\times n}$ is asymptotically stable and

$$f(x_{i-1}, u_i) \in \Re_+^n \text{ for } x_i \in \Re_+^n \text{ and } u_i \in \Re_+^m\,,\; i \in Z_+\,, \tag{22a}$$

$$g(x_i, u_i) \in \Re_+^p \text{ for } x_i \in \Re_+^n \text{ and } u_i \in \Re_+^m\,,\; i \in Z_+\,. \tag{22b}$$

Proof. *Sufficiency.* By Lemma 1 if $A \in \Re_+^{n\times n}$ is asymptotically stable then $A_1 \in \Re_+^{n\times n}$. It is well-known [13] that if $0 < \alpha \le 1$ then $c_j^{\alpha} > 0$ for $j = 1,2,...$. Therefore, from (19b) we have $\Phi_j \in \Re_+^{n\times n}$ for $j = 0,1,2,...$ and from (19a) $x_i \in \Re_+^n$ for $i = 1,2,...$ since by assumption (22a) $f_1(x_{i-1}, u_i) = A_1 f(x_{i-1}, u_i) \in \Re_+^n$ for $x_i \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$. If (22b) holds then from (14b) we have $y_i \in \Re_+^p$ for $i \in Z_+$.

Necessity. If $f(x_{i-1}, u_i) = 0$ then $x_i \in \Re_+^n$, $i \in Z_+$ only if $A_1 \in \Re_+^{n\times n}$ and by Lemma 1 implies the asymptotic stability of the matrix $A \in \Re_+^{n\times n}$. Note that $x_i \in \Re_+^n$ for $i \in Z_+$ implies the condition (22a). Similarly, $y_i \in \Re_+^p$ for $i \in Z_+$ implies the condition (22b).

Consider the fractional nonlinear system (14a) for zero inputs ($u_i = 0$ and $f(x_{i-1}, 0) = \bar{f}_2(x_{i-1})$ in the form

$$\Delta^{\alpha} x_i = Ax_i + \bar{f}_2(x_{i-1})\,,\; i \in Z_+\,,\; 0 < \alpha \le 1 \tag{23}$$

or

$$x_i = \sum_{j=1}^{i} A_1 c_j^{\alpha} x_{i-j} + f_2(x_{i-1}),\ i \in Z_+,\ 0 < \alpha \le 1, \tag{24a}$$

where

$$f_2(x_{i-1}) = A_1 \bar{f}_2(x_{i-1}),\ i \in Z_+ \tag{24b}$$

and A_1 is defined by (15c).

Definition 4. The positive fractional nonlinear system (23) is called asymptotically stable in the region $D \in \Re_+^n$ if $x_i \in \Re_+^n$, $i \in Z_+$ and

$$\lim_{i \to \infty} x_i = 0 \ \text{for}\ x_0 \in D \in \Re_+^n. \tag{25}$$

To test the asymptotic stability of the system the Lyapunov method will be used. As a candidate of the Lyapunov function we choose

$$V(x_i) = c^T x_i > 0 \ \text{for}\ x_i \in \Re_+^n,\ i \in Z_+, \tag{26}$$

where $c \in \Re_+^n$ is a vector with strictly positive components $c_i > 0$ for $i = 1,...,n$.
Using (26) and (24) we obtain

$$\begin{aligned}
\Delta V(x_i) &= V(x_{i+1}) - V(x_i) = c^T x_{i+1} - c^T x_i \\
&= c^T \left[\sum_{j=1}^{i+1} A_1 c_j^{\alpha} x_{i-j+1} + f_2(x_i) - \left(\sum_{j=1}^{i} A_1 c_j^{\alpha} x_{i-j} + f_2(x_{i-1}) \right) \right] \\
&= c^T \left[\sum_{j=1}^{i} A_1 c_j^{\alpha} (x_{i-j+1} - x_{i-j}) + A_1 c_{i+1}^{\alpha} x_0 + f_2(x_i) - f_2(x_{i-1}) \right] < 0
\end{aligned}$$

and

$$\sum_{j=1}^{i} A_1 c_j^{\alpha} (x_{i-j+1} - x_{i-j}) + A_1 c_{i+1}^{\alpha} x_0 + f_2(x_i) - f_2(x_{i-1}) < 0,\ x_i \in D \in \Re_+^n,\ i \in Z_+ \tag{27}$$

since $c \in \Re_+^n$ is strictly positive.
Therefore, the following theorem has been proved.

Theorem 5. The positive fractional discrete-time nonlinear system (23) is asymptotically stable in the region $D \in \Re_+^n$ if the condition (27) is satisfied.

Example 2. Consider the fractional discrete-time nonlinear system (23) with

$$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix},\ A = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{bmatrix},\ f_2(x_i) = \begin{bmatrix} x_{1,i} x_{2,i} \\ x_{2,i}^2 \end{bmatrix}. \tag{28}$$

In this case

$$A_1 = [I_2 - A]^{-1} = \begin{bmatrix} 0.7 & -0.1 \\ -0.2 & 0.6 \end{bmatrix}^{-1} = \frac{1}{0.4} \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix} \in \Re_+^{2 \times 2}.$$

The nonlinear system is positive since the matrix $A \in \Re_+^{2\times 2}$ is asymptotically stable and $f_2(x_i) \in \Re_+^2$ for all $x_i \in \Re_+^2$, $i \in Z_+$.
The region $D \in \Re_+^2$ is defined by

$$D := \{x_{1,i}, x_{2,i}\} = \sum_{j=1}^{i} A_1 c_j^{\alpha} x_{i-j+1} + A_1 c_{i+1}^{\alpha} x_0 - x_i + f_2(x_i) =$$

$$= \begin{bmatrix} 1.5\left(\sum_{j=1}^{i} c_j^{\alpha} x_{1,i-j+1} + c_{i+1}^{\alpha} x_{10}\right) + 0.25\left(\sum_{j=1}^{i} c_j^{\alpha} x_{2,i-j+1} + c_{i+1}^{\alpha} x_{20}\right) - x_{1,i} + x_{1,i}x_{2,i} \\ 0.5\left(\sum_{j=1}^{i} c_j^{\alpha} x_{1,i-j+1} + c_{i+1}^{\alpha} x_{10}\right) + 1.75\left(\sum_{j=1}^{i} c_j^{\alpha} x_{2,i-j+1} + c_{i+1}^{\alpha} x_{20}\right) - x_{2,i} + x_{2,i}^2 \end{bmatrix}. \tag{29}$$

Let us assume

$$x_{10} = 0.1, \; x_{20} = 0.2, \; \alpha = 0.5, \; i = 4. \tag{30}$$

The region defined by (29) with (30) is shown in Figure 2.

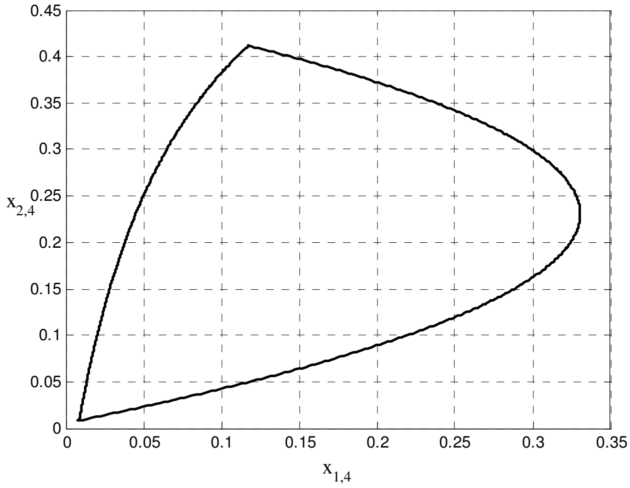

Fig 2. Stability region (inside the curved line).

Conclusions

The stability of fractional positive continuous-time and discrete-time nonlinear systems has been analyzed. Necessary and sufficient conditions for the positivity of the nonlinear systems have been given (Theorem 1 and 4). The well-known Lyapunov method has been extended to positive fractional nonlinear systems. Using this approach sufficient conditions for the asymptotic stability of fractional positive continuous-time nonlinear systems (Theorem 2) and discrete-time nonlinear systems (15) have been established. The considerations have been illustrated by numerical examples of fractional positive nonlinear system. An open problem is an extension of these considerations to descriptor fractional positive nonlinear systems.

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