Optimal Timing Control Using the Augmented Phase Reduction

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Summary. We develop an optimal control algorithm to change the period of a periodic orbit using a minimum energy input, which also minimizes the transversal distance to the uncontrolled periodic trajectory. Our algorithm uses a two-dimensional augmented phase reduction technique based on both isochrons and isostables. We show that our control algorithm is effective even when a large change in time period is required or when the nontrivial Floquet multiplier of the periodic orbit is close to one; in such cases, an analogous control algorithm based on standard phase reduction fails.

Augmented Phase Reduction

Periodic orbits are fundamentally important in dynamical systems theory, and they arise in many systems of physical, biological, and technological interest. The study of periodic orbits has benefitted greatly from the use of standard phase reduction based on isochrons, in which a single scalar phase variable captures the essence of an oscillation and its response to perturbations [1, 2, 3]. However, there are situations for which a recently-developed augmented phase reduction procedure [4] based on both isochrons and isostables can vastly improve the ability to understand and control the dynamics of a system with a periodic orbit. Consider the system

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}) + U(t), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $U(t) \in \mathbb{R}^n$ is an external perturbation. Standard phase reduction can be used to reduce this system to a one dimensional system for the phase $\theta \in [0, 2\pi)$ given by [3]:

$$\dot{\theta} = \omega + \mathcal{Z}(\theta)^T U(t). \tag{2}$$

For every nontrivial Floquet multiplier λ_i, with the corresponding eigenvector v_i, the set of isostable coordinates is defined as [4]

$$\psi_i(\mathbf{x}) = e_i^T V^{-1}(\mathbf{x}_\Gamma - \mathbf{x_0}) \exp(-\log(\lambda_i) t_\Gamma / T), \qquad i = 1, \ldots, n-1. \tag{3}$$

Here $\mathbf{x}_\Gamma$ and $t_\Gamma \in [0, T)$ are defined to be the position and the time at which the trajectory first returns to the isochron Γ_0, and e_i is a vector with 1 in the i^{th} position and 0 elsewhere. As shown in [4], we get the following equations for ψ_i and its gradient $\nabla_{\gamma(t)}\psi_i$ under the flow $\dot{\mathbf{x}} = F(\mathbf{x})$:

$$\begin{aligned} \dot{\psi}_i &= k_i \psi_i, & (4)\\ \frac{d\nabla_{\gamma(t)}\psi_i}{dt} &= \left(k_i I - DF(\gamma(t))^T\right) \nabla_{\gamma(t)}\psi_i, & (5) \end{aligned}$$

where $k_i = \log(\lambda_i)/T$, DF is the Jacobian of F, and I is the identity matrix. We refer to this gradient $\nabla_{\gamma(t)}\psi_i \equiv \mathcal{I}_i(\theta)$ as the *isostable response curve (IRC)*. Its T-periodicity along with the normalization condition $\nabla_{\mathbf{x_0}}\psi_i \cdot v_i = 1$ gives a unique IRC. It gives a measure of the effect of a control input in driving the trajectory away from the periodic orbit. It is often the case that only the isostable coordinate corresponding to the Floquet multiplier closest to the unit circle will need to be considered. Then we obtain the augmented phase reduction

$$\begin{aligned} \dot{\theta} &= \omega + \mathcal{Z}^T(\theta) \cdot U(t), & (6)\\ \dot{\psi} &= k\psi + \mathcal{I}^T(\theta) \cdot U(t), & (7) \end{aligned}$$

where Z is the (infinitesimal) phase response curve. Since we are only considering one isostable coordinate, we have removed the subscript for ψ. Ignoring the ψ equation gives the standard phase reduction.

Optimal Timing Control for Hopf Bifurcation Normal Form

An optimal control law based on the augmented phase reduction to change the period of a periodic orbit is found by using the cost function $C[u(t)]$:

$$C[u(t)] = \int_0^{T_1} \left[\alpha u^2 + \beta\psi^2 + \lambda_1\left(\dot{\theta} - \omega - \mathcal{Z}(\theta)u(t)\right) + \lambda_2\left(\dot{\psi} - k\psi - \mathcal{I}(\theta)u(t)\right)\right] dt. \tag{8}$$

The first term in the cost function ensures that the control law uses a minimum energy input, and the second term minimizes the transversal distance (in the direction of the slow isostable coordinate ψ) from the uncontrolled periodic trajectory. The last two terms ensure that the system obeys the augmented phase reduction, with λ_1 and λ_2 being the Lagrange

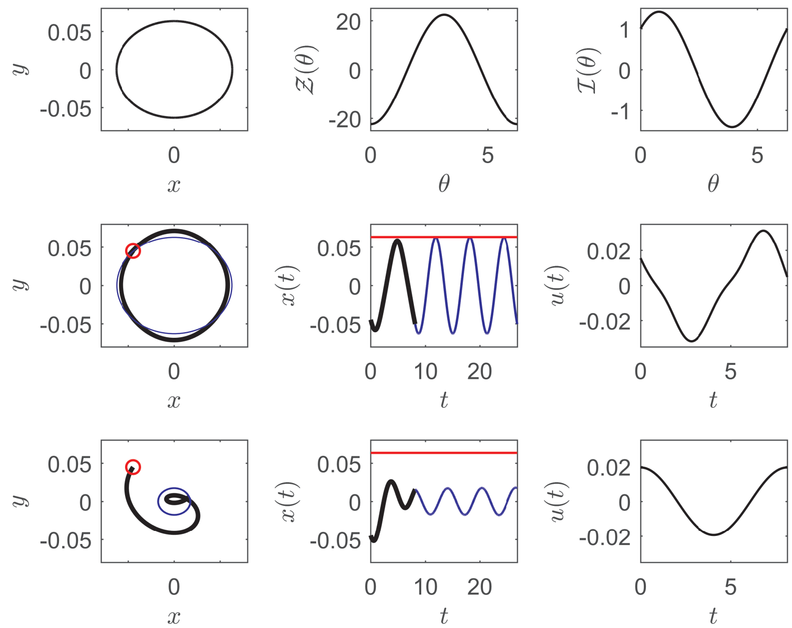

Figure 1: Top row shows the uncontrolled periodic orbit, PRC, and IRC for the Hopf normal form with parameters given in the main text. The middle (resp., bottom) row shows the trajectory, time series, and control input for control based on the augmented (resp., standard) phase reduction. Control is on (resp., off) for the portion shown by the thick black (resp., thin blue) line. The trajectory starts at the small red circle. The red horizontal line shows the amplitude of the uncontrolled periodic orbit.

multipliers. The resulting Euler-Lagrange equations are solved as a two point boundary value problem with the boundary conditions

$$\theta(0) = 0, \qquad \theta(T_1) = 2\pi, \qquad \psi(0) = 0, \qquad \psi(T_1) = 0. \tag{9}$$

The last boundary condition makes sure that trajectory ends back on the periodic orbit. The corresponding optimal control problem with standard phase reduction [5] can be obtained by setting $\beta = 0$ and $\lambda_2 = 0$ in the cost function.
We use our control algorithm to change the period of a periodic orbit near the supercritical Hopf bifurcation. The normal form of the supercritical Hopf bifurcation with an external control input $u(t)$ is:

$$\dot{x} = ax - by + (x^2 + y^2)(cx - dy) + u(t), \tag{10}$$
$$\dot{y} = bx + ay + (x^2 + y^2)(dx + cy). \tag{11}$$

With zero control input $u(t)$, $c < 0$, and $a < 0$, the system has a stable fixed point. As a increases through 0, a stable periodic orbit is born, and the fixed point becomes unstable. With parameters $a = 0.004, b = 1, c = -1, d = 1$, the system has a stable periodic orbit with the time period $T = 6.2582$ and the nontrivial Floquet multiplier $\exp(-2aT) = 0.9512$. The PRC and the IRC are sinusodial with amplitudes $\sqrt{\frac{d^2+c^2}{-ac}}$ and $\sqrt{1 + \frac{d^2}{c^2}}$, respectively. The top row of Figure 1 shows the uncontrolled periodic orbit, PRC, and IRC for the given parameter values. The control parameters α and β are taken to be unity. We calculate the optimal control with $T_1 = 1.3T = 8.1356$ both for the augmented and standard phase reduction. The resulting trajectories, time series, and control inputs are shown in the bottom two rows of Figure 1. As seen in this figure, the control based on the augmented phase reduction does much better in changing the period of the periodic orbit while also keeping the trajectory close to the periodic orbit for the uncontrolled system. A parametric study shows that the augmented phase reduction based control is much more effective than the standard phase reduction based control, especially when the desired change in period is large and/or the nontrivial Floquet multiplier of the periodic orbit is close to 1. A similar approach for other systems, including higher-dimensional systems, shows comparable results.

References

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