Long-term stochastic stability of locally stable dynamical systems with respect to white noise

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Summary. The effect of small stochastic perturbations on a dynamical system with locally stable fixed point is considered. The perturbed system is described by the Ito stochastic differential equations such that the noise does not vanish at the equilibrium. It is known that in this case almost all trajectories escape from any bounded domain and the stability with respect to such perturbations on an infinite time interval holds for a relatively narrow class of globally stable dynamical systems. We describe the classes of perturbations such that the stability of the locally stable equilibrium holds for polynomially and exponentially long time intervals with respect to a small perturbation parameter.

Problem statement

Consider the system of ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x},t), \quad t \geq 0, \quad \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

Let $\mathbf{x} = 0$ be an equilibrium, $\mathbf{f}(0,t) \equiv 0$. Suppose the vector-valued function $\mathbf{f}(\mathbf{x},t) = (f_1(\mathbf{x},t),\dots,f_n(\mathbf{x},t))$ is continuous, satisfies a Lipschitz condition: $|\mathbf{f}(\mathbf{x},t)-\mathbf{f}(\mathbf{y},t)| \leq L|\mathbf{x}-\mathbf{y}|$ and a growth condition: $|\mathbf{f}(\mathbf{x},t)| \leq M(1+|\mathbf{x}|)$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, $t \geq 0$ with positive constants L, M. Assume that there exists a local Lyapunov function $V(\mathbf{x},t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R})$ satisfying the following properties:

$$\begin{gathered} |\mathbf{x}|^2 \leq V(\mathbf{x},t) \leq A|\mathbf{x}|^2, \quad |\partial_{\mathbf{x}} V|^2 \leq B|\mathbf{x}|^2, \quad |\partial_{x_i}\partial_{x_j} V| \leq C, \\ \frac{dV}{dt}\Big|_{(1)} \stackrel{def}{=} \frac{\partial V}{\partial t} + \sum_{k=1}^{n} \frac{\partial V}{\partial x_k} f_k \leq -\gamma V \quad \forall\, |\mathbf{x}| \leq r_0, \quad t \geq 0, \end{gathered} \tag{2}$$

with constants $A, B, C, \gamma, r_0 > 0$. This implies that the solution $\mathbf{x}(t) \equiv 0$ is locally asymptotically stable and does not exclude the existence of any other stable fixed points. Note that such local Lyapunov functions are usually constructed for nonlinear differential equations (see, for instance, [1]). Together with (1) we consider the Itô stochastic differential equation:

$$d\mathbf{z}(t) = \mathbf{f}(\mathbf{z},t)\,dt + \mu\, G(\mathbf{z},t)\,d\mathbf{w}(t), \quad \mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{R}^n. \tag{3}$$

Here $\mathbf{w}(t) = (w_1(t),\dots,w_n(t))$ is n-dimensional Wiener process defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, $G(\mathbf{z},t) = \{g_{ij}(\mathbf{z},t)\}$ is a given continuous $n\times n$ matrix, satisfies the Lipschitz and the growth conditions. A small positive parameter $0 < \mu \ll 1$ controls the intensity of the perturbation. The matrix $G(\mathbf{z},t)$ and the initial data $\mathbf{z}_0$ are assumed to be deterministic. These constraints are sufficient for the existence and uniqueness of a global solution to the initial value problem (3) for all $\mathbf{z}_0 \in \mathbb{R}^n$ (see [2]). In addition, suppose that the trivial solution $\mathbf{z}(t) \equiv 0$ is not preserved in the perturbed system, namely, $G(0,t) \not\equiv 0$. Our goal is to describe a class of matrices G guaranteing the stochastic stability of the trivial solution to system (1) under a persistent perturbation by white noise.

Many authors have considered the stability of stochastic differential equations with $G(0,t) \equiv 0$. However, the problem with persistent perturbations ($G(0,t) \not\equiv 0$) was investigated only in several works [3, 4, 5]. First, from [3] it follows that there is no stochastic stability in autonomous systems and almost all perturbed trajectories escape from any bounded domain. In [4, §7.4], Khasminskii proved that if $\|G(\mathbf{z},t)\|$ decays sufficiently fast at infinity $t \to \infty$ and the unperturbed system is globally stable, the stability of solution $\mathbf{x}(t) \equiv 0$ is preserved for $t > 0$. The existence of a global Lyapunov function is sufficient for stochastic stability with respect to white noise with uniformly bounded matrix G (see [4, §7.7]). However, if the dynamical system (1) has at least one more stable fixed point $\mathbf{x}_* \neq 0$, the perturbation with uniformly bounded matrix G leads to the loss of stability of $\mathbf{x}(t) \equiv 0$. Thus, the stability under persistent perturbation by white noise occurs in a fairly narrow class of systems, and it is therefore worth while to consider the problem of stability on a finite time interval (see [5, 3]). One variant of such approach is to find the largest possible time interval $[0;T_\mu]$ on which solutions to the perturbed equation are close to the equilibrium of the deterministic system. It was shown in [5, Ch. 9] that if the matrix G is uniformly bounded and there exists a local Lyapunov function having properties (2) with $\gamma = 0$, then the equilibrium is strongly stable on the interval $0 \leq t \leq \mathcal{O}(\mu^{-2})$. The stability of solution on longer time intervals remains an open question.

We show that if $\|G(\mathbf{z},t)\| = \mathcal{O}(1)$ for all $t \geq 0$ and $|\mathbf{z}| \leq r_0$, then the locally asymptotically stable solution $\mathbf{x}(t) \equiv 0$ to the deterministic system is stochastically stable with respect to white noise on the asymptotically long time interval $0 \leq t \leq \mathcal{O}(\mu^{-N})$ for any $N \geq 2$. We also prove that if $\|G(\mathbf{z},t)\| = \mathcal{O}(t^{-1/2})$ as $t \to \infty$ for all $|\mathbf{z}| \leq r_0$, the stochastic stability holds for $0 \leq t \leq \mathcal{O}(\exp\mu^{-1})$.

Definition of stability

Let us consider the definition of stochastic stability that will be used below.
Definition. *The solution* $\mathbf{x}(t) \equiv 0$ *to system* (1) *is said to be stochastically stable with respect to white noise on the time interval* $[0; T_\mu]$ *uniformly for G from the set* $\mathcal{P}$*, if*

$$\forall\, \varepsilon, \nu > 0 \;\; \exists\, \delta, \Delta > 0 : \;\; \forall\, |\mathbf{z}_0| < \delta, \;\; \mu < \Delta, \;\; G \in \mathcal{P}$$

the solution $\mathbf{z}(t)$ *to the initial value problem* (3) *satisfies the inequality:*

$$\mathbf{P}\Big(\sup_{t \in [0; T_\mu]} |\mathbf{z}(t)| \geq \varepsilon \Big) \leq \nu.$$

Thus, if the trivial solution is stochastically stable on the time interval $[0; T_\mu]$, the perturbed trajectories $\mathbf{z}(t)$ with sufficiently small initial values and perturbation parameter do not leave an arbitrary small neighbourhood of zero with probability tending to one at least for $t \leq T_\mu$.

Class of perturbations

For every $\alpha \geq 0$, we define the class $\mathcal{A}^\alpha$ as a set of matrices $G(\mathbf{z}, t)$ such that

$$\sup_{|\mathbf{z}| \leq r_0, t \geq 0} \|G(\mathbf{z}, t)\|(1 + t)^\alpha < \infty.$$

By $\mathcal{A}^\alpha_h$ we denote the set of matrices $G \in \mathcal{A}^\alpha$ such that $\|G(\mathbf{z}, t)\|(1 + t)^\alpha \leq h$ for all $|\mathbf{z}| \leq r_0$ and $t \geq 0$.

Main results

Theorem 1. *Suppose that unperturbed system* (1) *has a Lyapunov function* $V(\mathbf{x}, t)$ *satisfying* (2)*. Then for all* $N \in \mathbb{N}$*,* $h > 0$*, and* $0 < \kappa < 1$ *the solution* $\mathbf{x}(t) \equiv 0$ *to system* (1) *is stochastically stable with respect to white noise for* $0 \leq t \leq \mu^{-2N+\kappa}$ *uniformly for* $G \in \mathcal{A}^0_h$.

The interval of stability with respect to white noise can be increased by decreasing the coefficients of the perturbation matrix. We have
Theorem 2. *Suppose that unperturbed system* (1) *has a Lyapunov function* $V(\mathbf{x}, t)$ *satisfying* (2)*. Then for all* $h > 0$ *and* $0 < \kappa < 1$ *the solution* $\mathbf{x}(t) \equiv 0$ *to system* (1) *is stochastically stable with respect to white noise for* $0 \leq t \leq \exp \mu^{-2+\kappa} - 1$ *uniformly for* $G \in \mathcal{A}^{1/2}_h$.

The stability is proved by the method of stochastic Lyapunov functions (see [6]).

References

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