On the Analysis of Quasi-periodic Systems and a Novel "Deterministic" Explanation of the Stochastic Resonance Phenomenon

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Summary. The present paper is concerned with the analysis of quasi-periodic systems which represent an important class of dynamical systems frequently encountered in applications. The related problems appear, e.g. in the analysis of parametric amplifiers based on resonant micro- and nanoscale systems, wave propagation in nearly periodic structures and composite materials, etc. The conventional methods of linear and nonlinear dynamics, e.g. asymptotic methods and approximate approaches based on Floquet theory, often appear to be not applicable for solving such problems. The present paper addresses the application of the Oscillatory Strobodynamics approach and the Method of Varying Amplitudes for these problems with the notable effects being revealed. In particular, a novel "deterministic" explanation to the phenomenon of stochastic resonance is proposed.

Introduction

Systems involving combined multiple parametric and direct excitations with aliquant frequencies relate to an important class of dynamical systems, the quasi-periodic systems. The corresponding problems appear in many fields of science and technology, e.g. in the analysis of the dynamics of parametric amplifiers based on resonant micro- and nanoscale electromechanical systems, linear and nonlinear wave propagation in nearly periodic structures and composite materials, etc. The conventional methods of linear and nonlinear dynamics often appear to be not applicable for solving such problems due to the restrictions they impose on the problem parameter space. In the present paper, two novel analytical approaches are proposed to be used for studying the considered class of systems. The first one is the Oscillatory Strobodynamics (OS) approach [1,2]. The name is motivated by the fact that the considered system behavior is perceived as under a stroboscopic light so that only the main, slow component of motions is captured. In particular, by means of the OS approach, the results are obtained that provide a novel "deterministic" explanation to the well-known phenomenon of stochastic resonance [3]. Another approach also employed in the paper is the Method of Varying Amplitudes (MVA) [4,5]. This approach is strongly related to the methods based on Floquet theory and may be also considered as a continuation of the classical methods of harmonic balance and averaging. It implies representing a solution in the form of a harmonic series with varying amplitudes; however, in contrast to the asymptotic methods, the amplitudes are not required to vary slowly. Thus the MVA does not assume the presence of a small parameter in the governing equations.

Governing equations

As an example of the approaches application, consider the following general equation that describes motions of an oscillator under combined multiple parametric and direct excitations:

$$\ddot{x}+\beta\dot{x}+\lambda^2(1+p_1\cos\Omega_{\mathrm{p1}}t+p_2\cos(\Omega_{\mathrm{p2}}t+\gamma))x+kx^3=d_1\cos(\Omega_{\mathrm{d1}}t+\phi)+d_2\cos(\Omega_{\mathrm{d2}}t+\delta)\ . \tag{1}$$

For $\beta=p_2=k=d_1=d_2=0$ equation (1) reduces to the classical Mathieu equation, for $p_1=p_2=d_2=0$ we get the classical Duffing equation, and for $p_2=d_2=0$, $\Omega_{\mathrm{p1}}=2\Omega_{\mathrm{d1}}$ equation (1) describes the dynamics of a nonlinear parametric amplifier [6,7]. First, by means of the OS approach, the case of combined low- and high-frequency excitations is considered, when $\Omega_{\mathrm{d2}}=\Omega_{\mathrm{p2}}=\Omega$, $\Omega_{\mathrm{p1}}=\omega_p,\Omega_{\mathrm{d1}}=\omega_d$, and $\Omega\gg\omega_p,\omega_d,\lambda$:

$$\ddot{x}+\beta\dot{x}+\lambda^2(1+p_1\cos\omega_pt+p_2\cos(\Omega t+\gamma))x+kx^3=d_1\cos(\omega_dt+\phi)+d_2\cos(\Omega t+\delta)\ . \tag{2}$$

Then the MVA is used to study the case $p_2=d_2=0$, without any restrictions on the frequencies $\Omega_{\mathrm{p1}}=\Omega_{\mathrm{p}}$ and $\Omega_{\mathrm{d1}}=\Omega_{\mathrm{d}}$ being imposed:

$$\ddot{x}+\beta\dot{x}+\lambda^2(1+p_1\cos\Omega_{\mathrm{p}}t)x+kx^3=d_1\cos(\Omega_{\mathrm{d}}t+\phi)\ . \tag{3}$$

Equation (3) describes the response of a nonlinear parametric amplifier when there is a detuning between direct and parametric excitation frequencies. Such frequency detuning can be present in real micro- and nanoscale electromechanical devices due to manufacturing and operating imperfections and nonlinear factors. Moreover, the two to one relation between parametric and direct excitation frequencies is optimal only for the linear amplifier; for the nonlinear one, the optimal relation between frequencies is yet to be determined.

The case $d_1=d_2=0$ is also studied by means of the MVA to reveal general effects of combined parametric excitations with aliquant frequencies on the system response:

$$\ddot{x}+\beta\dot{x}+\lambda^2(1+p_1\cos\Omega_{\mathrm{p1}}t+p_2\cos(\Omega_{\mathrm{p2}}t+\gamma))x+kx^3=0\ . \tag{4}$$

Results obtained

Employing the OS approach and the method of direct separation of motions [2] for solving equation (2), one obtains equation describing only slow motions of the oscillator (the Vibro-transformed Dynamics Equation [2]). From this equation it follows that external and parametric resonances can occur in the system both by shifting the low frequencies ω_p, ω_d with the fast frequency Ω being fixed and by shifting the fast frequency with the low ones being fixed. It is also shown that by varying the amplitudes of the high-frequency excitations p_2 and d_2 it is possible to attain resonance conditions without changing other parameters, see e.g. Figure 1(a) where the dependency of the maximum amplitude of the system response a on the parametric excitation amplitude p_2 is shown schematically. These results (see also [8]) provide a novel "deterministic" explanation to the well-known phenomenon of stochastic resonance [3] which manifests itself in the nonmonotonic change of the system response with increasing the intensity of random excitation.

Equation (3) features not periodic, but quasi-periodic stationary solutions which were determined by means of the MVA. As an illustration, Figures 1(b), (c) show the dependencies of the maximum amplitude of the system steady-state response a on the external excitation frequency Ω_d for $p = 0.35, k = 1$, and (b) $\Omega_p = 2.05\Omega_d$, (c) $\Omega_p = 1.95\Omega_d$. Solid lines represent stable solutions, dotted lines unstable ones, crosses represent results obtained by direct numerical integration of equation (3) by Wolfram Mathematica 7.

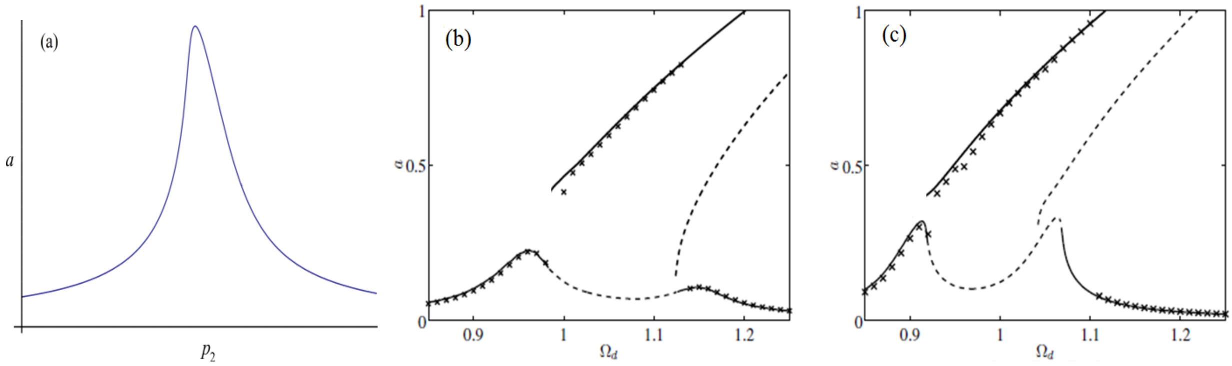

Figure 1. The dependencies of the maximum amplitude of the system steady-state response a on the system parameters.

The results obtained by means of the MVA indicate that maximum amplitude of the amplifier steady-state response can be larger in the detuned case than in the perfectly tuned one. So that detuning between parametric and direct excitation frequencies can be introduced purposefully to improve characteristics of the system. The time-averaged amplitude of the amplifier response, however, is always larger in the perfectly tuned case.

Similarly to equation (3), stationary solutions of equation (4) are not periodic, but quasi-periodic. These were determined by means of the MVA and compared with the results of direct numerical integration of (4) showing good agreement.

Conclusions

The paper illustrates the application of the OS approach and the MVA for the analysis of quasi-periodic systems that involve combined multiple parametric and direct excitations with aliquant frequencies. First, the case of combined low- and high-frequency excitations is studied by means of the OS approach. As the result, a novel "deterministic" explanation to the phenomenon of stochastic resonance is proposed. Then the response of a nonlinear parametric amplifier is studied for the case when there is a detuning between direct and parametric excitation frequencies. Employing the MVA, it is shown that this frequency detuning can be beneficial for the amplifier. General effects of combined parametric excitations with aliquant frequencies are also considered. The obtained results are relevant for micro- and nano- electromechanical systems and facilitate analysis of wave propagation in nearly periodic structures and composite materials.

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