Generic Bifurcations at Nonlocal Continua Described by Fractional Calculus

Péter B. Béda*
**Department of Vehicle Elements and Vehicle Structure Analysis, Budapest University of Technology and Economics, Budapest, Hungary*

Summary. In material instability several problems appear in numerical post-localization analysis as a consequence of non-generic behavior. Conventionally, non-local properties are included into the constitutive equations in form of strain gradient dependent terms. In case of second gradient dependence a generic static bifurcation can easily be recognized as material instability phenomenon. Another way to include non-locality is the use of fractional calculus. By introducing fractional derivatives a generalized strain can be defined. Then both dynamic and static bifurcations can be generic. When the fractional order goes to an integer, the classical results are obtained again.

Generic bifurcation and constitutive equation

Non-local materials were already studied in the 1960s by several authors (for example [1]) as a part of continuum mechanics. When material instability gained more interest, non-local behavior appeared again [2], because instability zones exhibited singular properties for local constitutive equations. Such works used the gradient of strain tensors to include non-locality into the constitutive equation Most gradient theories concentrate on the second gradient

$$\dot{\sigma} = c_1\dot{\varepsilon} + c_2\ddot{\varepsilon} - c_3\frac{\partial^2\dot{\varepsilon}}{\partial x^2} \tag{1}$$

then to form the set of the basic equations of continua the equation of motion and the kinematic equation should be added. By transforming them into the velocity field and using new variables $y_1 = v, y_2 = \dot{v}$ a dynamical system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)\dot{y}_1 + c_2\frac{\partial^2}{\partial x^2}y_2, \tag{2}$$

is obtained. Its characterisic equation for λ reads

$$\lambda^2 y_1 - \lambda c_2\frac{\partial^2}{\partial x^2}y_1 - \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)y_1 = 0. \tag{3}$$

The critical eigenfunction of (2) at the loss of stability ($c_1 = c_{1crit} < 0$) is $y_1 = \exp\left(ix\frac{\pi}{\ell^*}\right)$, where $\ell^* := \pi\sqrt{-\frac{c_3}{c_{1crit}}}$ can be identified as (static) internal length. Bifurcation analysis should be performed on the nontrivial eigenspace of the critical eigenvalues. Thus necessary condition for a generic bifurcation is to have regular nontrivial eigenspace. In case of both rate and gradient independent ($c_2 = c_3 = 0$) constitutive equations no generic bifurcation is possible.

The use of fractional continuum mechanics

Following the idea of [3] strain can be generalized to fractional derivatives

$$\sigma = \tilde{c}_1\frac{1}{2}({}^C D^\alpha_{a+}u(x) - {}^C D^\alpha_{L-}u(x)), \tag{4}$$

where ${}^C D^\alpha_{a+}u(x)$ and ${}^C D^\alpha_{L-}u(x)$ are α-th Caputo fractional derivatives with respect to x,

$$({}^C D^\alpha_{a+}u)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{u(\xi)-u(a)}{(x-\xi)^\alpha}d\xi.$$

In (4) non-locality is obviously added by the use of fractional derivative: value of the derivatives of displacement u at position x is determined by all values before and after that position. In the following two cases a fractional derivative generalization of strain is used as $\varepsilon = \frac{d^\alpha u}{dx^\alpha}$.

Static bifurcation

For constitutive equation a "classical" second gradient derivative dependent one is used like (1). Its rate form reads

$$\dot{\sigma} = B\dot{\varepsilon} + C\frac{\partial^2}{\partial x^2}\dot{\varepsilon}.$$

Now the equation in the velocity field is

$$\rho\ddot{v} = B\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial}{\partial x}v + C\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^3}{\partial x^3}v. \tag{5}$$

For periodic perturbations the characteristic equation of (5) has zero solution, if

$$(B-\omega^2 C)\frac{\partial^\alpha}{\partial x^\alpha}e^{i\omega x}=0. \quad \text{Thus} \quad \frac{B}{C}=\omega^2 \Rightarrow \omega=\sqrt{\frac{B}{C}}$$

and the non-trivial eigenfunctons are of form $y_1 = e^{i\sqrt{\frac{B}{C}}x}$. That is, the condition for the existence of a generic static bifurcation and static internal length are the same for both fractional and conventional strain tensors.

Dynamic bifurcation

Now we study Malvern-Cristescu type constitutive equation in form

$$\sigma + D\dot{\sigma} = E\varepsilon + H\dot{\varepsilon}.$$

Now the characteristic equation for periodic perturbations reads

$$-\lambda^2\left(\lambda+\frac{1}{D}\right)+\lambda\frac{H}{D}\omega^\alpha e^{i\frac{\pi}{2}\alpha}+\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}e^{i\frac{\pi}{2}(1+\alpha)}=0. \tag{6}$$

Assume that the system undergoes a dynamic bifurcation ($Re\lambda = 0$) then $\lambda = i\beta$ and (6) has the form

$$i\beta^3+\frac{1}{D}\beta^2+\left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}+\beta\frac{H}{D}\omega^\alpha\right)e^{i\frac{\pi}{2}(1+\alpha)}=0. \tag{7}$$

Calculating the real and imaginary parts of (7) we have

$$-\frac{1}{D}\beta^2+\left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}+\beta\frac{H}{D}\omega^\alpha\right)\sin\frac{\pi}{2}\alpha=0 \tag{8}$$

and

$$\beta^3+\left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}+\beta\frac{H}{D}\omega^\alpha\right)\cos\frac{\pi}{2}\alpha=0. \tag{9}$$

From (8) β can be expressed

$$\beta_{12}=\frac{-H\sin\left(\frac{\pi}{2}\alpha\right)\omega^\alpha \pm \sqrt{H^2\sin^2\left(\frac{\pi}{2}\alpha\right)\omega^{2\alpha}+4\frac{E}{\rho}\sin\left(\frac{\pi}{2}\alpha\right)\omega^{1+\alpha}}}{-2}.$$

Then β should be substituted into (9) and an equation consisting of the value α, the material parameters E, D, H, ρ and perturbation frequency ω is obtained. For the sake of simplicity assume that $H = 0$, then this equation is

$$\left(\frac{E}{\rho}\sin\left(\frac{\pi}{2}\alpha\right)\omega^{1+\alpha}\right)^{\frac{3}{2}}+\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}\cos\left(\frac{\pi}{2}\alpha\right)=0. \tag{10}$$

When nontrivial solutions of (10) are searched for

$$\omega_c=\left(\frac{\cos\frac{\pi}{2}\alpha}{D}\frac{1}{\sqrt{\frac{E}{\rho}\sin\frac{\pi}{2}\alpha}}\right)^{\frac{2}{1+\alpha}} \tag{11}$$

is obtained. From (11) an ω_c can be calculated, which is the "critical" perturbing frequency for non-trivial eigenfunction $y_1 = \exp(ix\omega_c)$ and a dynamic internal length can be defined $\ell^d := \pi\omega_c$. In (11) we can see how D and $0 < \alpha < 1$ acts on "critical" perturbing frequency (and consequently on existence of generic dynamic bifurcation).

Conclusions

By using fractional derivatives in generalization of strain and periodic perturbations material stability analysis was performed for two types of constitutive equations. For a second gradient dependent constitutive equation the necessary conditions for a generic static bifurcation are the same for both fractional and conventional strains. For Malvern-Cristescu equation with fractional strain a dynamic internal length can be defined. The existence of a regular (nonzero an finite) dynamic internal length is necessary for generic dynamic bifurcation. The values of the critical perturbing frequency and dynamic internal length decreases as the order of the derivative in strain or the stress rate intensity factor (D) increases. Such dynamic internal length disappears when the order of the derivative in strain is one and then no generic dynamic bifurcation is possible (the same as the classical result).

References

[1] Mindlin, R.D. (1965) Second gradient of strain and surface tension in linear elasticity, *Intl. Journal of Solids and Structures*, **1**:417–438.

[2] Aifantis. E.C. (1992) On the role of gradients in the localization of deformation and fracture. *International Journal of Engineering Science*, **30**:1279–1299,

[3] Atanackovic T. M. B. Stankovic. (2009) Generalized wave equation in nonlocal elasticity, *Acta Mech* **208**:1–10