Forced and damped solitons in cyclic and symmetric structures

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Summary. The effect of modulationally unstable plane waves in a non-conservative cyclic and symmetric system is investigated. The main application is on vibration of bladed-disks, which are usually composed of a cyclic and periodic structure where non-linearities arise e.g. due to friction dissipation and large deformations. In this paper, we focus on unstable plane waves and their corresponding localised states through envelope dynamics. The results suggest that cyclic and symmetric non-conservative structures may experience very strong localised vibrations resembling envelope solitons if initial conditions are selected properly.

Introduction

Localisation of vibrations is a very active research field in structural dynamics, e.g. due to its importance for high-cycle fatigue. In the linear regime, localised vibrations arise due to manufacturing variability, and the aerospace engineering community usually refers to this phenomenon as mistuning. However, recent investigations have shown that even perfectly symmetric structures can localise vibrations if they are operating in the non-linear regime. Moreover, it is well accepted that rotating machines can experience travelling wave states, e. g. due to aeroelastic excitation and Coriolis effects. Therefore, this research focus on travelling waves and their corresponding non-linear modulation. Initial studies focused on a minimal cyclic and symmetric conservative model, and it has been demonstrated [1] that solitons can emerge if initial conditions are selected properly. Within this investigation we demonstrate that solitons can still emerge in the non-conservative case if an external force is added in order to balance the dissipation.

Physical System

The proposed minimal model is presented in Fig. 1. It consists of N_s identical masses m cyclically connected through

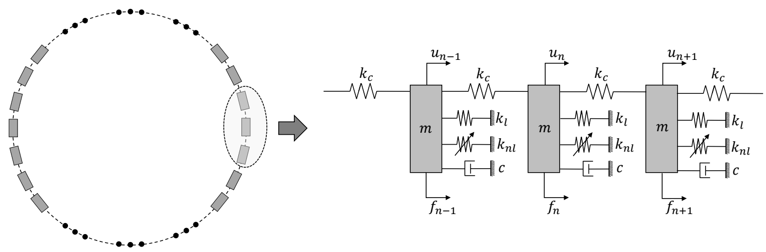

Figure 1: Proposed mechanical system. In the left hand side, an illustration of the full system, while the right hand side displays any three neighbour degrees of freedom.

linear springs k_c and externally excited by a force f. Each mass is also connected to the ground by a linear spring k_l, a viscous damper c, and a non-linear spring k_{nl} of cubic behaviour. The mathematical model for the n-th mass in Fig. 1 is written in the form

$$\ddot{u}_n + \Gamma\dot{u}_n + \omega_0^2 u_n + \xi u_n^3 - \omega_c^2(u_{n-1} + u_{n+1} - 2u_n) = h_n, \tag{1}$$

where u_n is the n-th displacement, $\Gamma = c/m, \omega_0^2 = k_l/m, \omega_c^2 = k_c/m, \xi = k_{nl}/m$, and $h_n = f_n/m$. Firstly, the external force is assumed as a travelling wave excitation such that

$$h_n(t) = h\exp\{i[k(n-1)a - \omega_F t]\} + c.c \tag{2}$$

where h denotes the force amplitude, ω_F states its frequency, k is the wave number, $a = 2\pi/N_s$ is the lattice parameter, t is the temporal coordinate, while $c.c$ is the complex conjugate of the first expression. The solution of Eq. 1 is written as a travelling wave modulated by a slowly-varying envelope function Ψ as

$$u_n(t) = \epsilon\Psi(X,T)\exp\{i[k(n-1)a - \omega_k t]\} + c.c, \tag{3}$$

where $X = \epsilon x$ and $T = \epsilon t$ are continuum spatial and time domains, respectively, $\omega_k = \sqrt{\omega_0^2 + 2\omega_c^2(1-\cos(ka))}$ is the linear dispersion relation, while ϵ is a small parameter. After inserting Eq. 3 into Eq. 1, the following expression

$$i\frac{\partial\Psi}{\partial\tau} + P\frac{\partial^2\Psi}{\partial\eta^2} + Q|\Psi|^2\Psi = -iA\Psi - B\exp\{i\delta\omega\tau\}, \tag{4}$$

is obtained if: (1) the solution is assumed in the continuum approximation; (2) only terms up to ϵ^3 are considered; (3) the damping is small and up to order ϵ^2 ($\Gamma \sim \epsilon^2\Gamma$); and (4) the driving force amplitude is also small and compared to order ϵ^3 ($h \sim \epsilon^3 h$). In Eq. 4, the parameter $P = \frac{1}{2}\frac{d^2\omega_k}{dk^2}$ reflects the group velocity dispersion, $Q = -\frac{3\xi}{2\omega_k}$ is the non-linear term, $A = \frac{\Gamma}{2}$ states the damping effect, and $B = \frac{h}{2\omega_k}$ indicates the external force, $\delta\omega = \omega_k - \omega_F$ is the detuning parameter, while $\tau = \epsilon^2 t$ is the slowly-varying time coordinate and $\eta = X - c_g T$ is the spatial variable in a frame moving with the group velocity $c_g = \frac{d\omega_k}{dk}$. Equation 4 is known in the literature as the ac-driven and damped Non-Linear Schrödinger Equation (NLSE), and its investigation is still an active research field in applied mathematics and theoretical physics.
Firstly, the solution of Eq. 4 is assumed to be locked in the driver's frequency as $\Psi_s(\eta,\tau) = \Psi_0(\eta)\exp\{i\delta\omega\tau\}$. Moreover, $\delta\omega$ is assumed negative such as a new parameter $v = -\delta\omega$ has strictly positive values. After inserting Ψ_s into Eq. 4 and introducing a new change of variables such as $\phi(y) = -\sqrt{-Q/(2v)} \times \Psi_0(\eta)$ and $y = \sqrt{-v/P} \times \eta$, the following autonomous system is obtained [2]

$$\frac{\partial^2\phi}{\partial y^2} - \phi + 2|\phi|^2\phi = iA_{eff} - B_{eff} \tag{5}$$

where $A_{eff} = A/v$ and $B_{eff} = (B/v) \times \sqrt{-Q/(2v)}$. In this paper, we solve Eq. 5 based on a finite difference approach coupled with a stability analysis presented in Ref. [2] .

Results

The system depicted in Fig. 1 is numerically investigated by assuming N_s=36, $\omega_0^2 = 1$ s^{-2}, $\omega_c^2 = 1$ s^{-2}, and $\xi = 0.1$ m^{-2}s^{-2}. The travelling wave excitation is assumed at wave number $k = 16$ and $\omega_F = 2.23$ rad/s, therefore $\delta_\omega =\sim -2.11 \times 10^{-2}$ rad/s. The existence and stability chart for the soliton solution in Eq. 5, assuming periodic boundary conditions, is presented in left-hand side of Fig. 2. The results are, basically, stable for low $|B_{eff}|$ and unstable for high

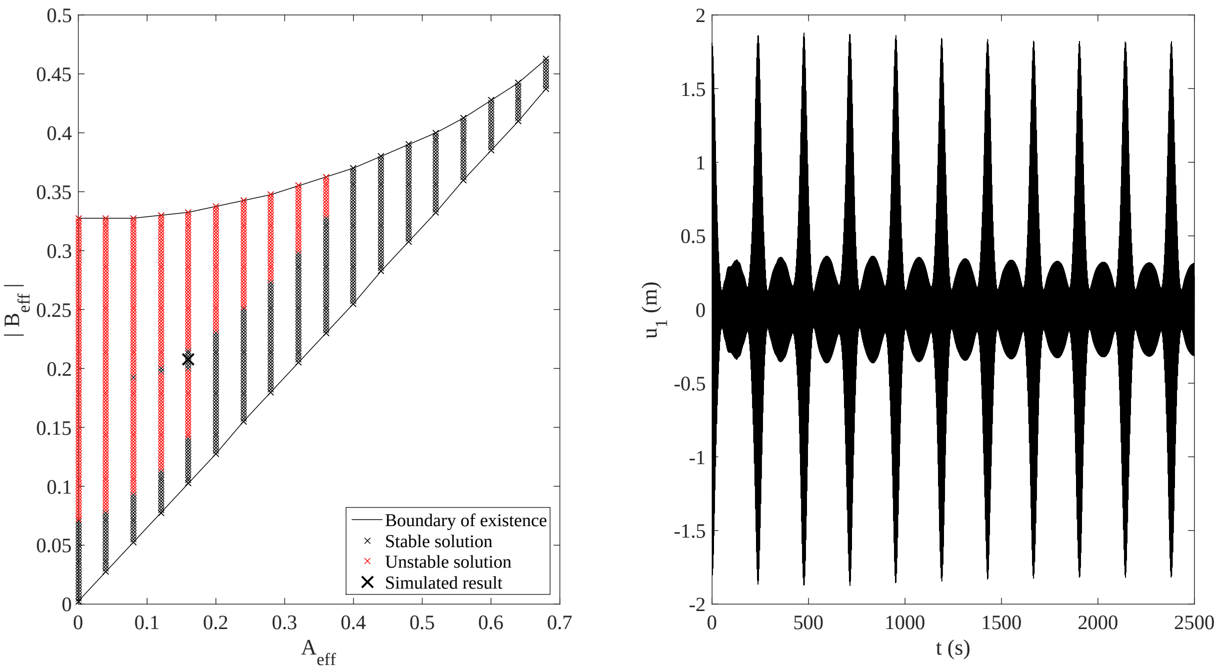

Figure 2: Left: Stability chart for Eq. 5. Right: Simulated results obtained from numerical integration of Eq. 1 using initial conditions from the stability chart.

$|B_{eff}|$ when $A_{eff} \lesssim 0.4$. However, a stability window is observed for A_{eff}=0.16, as already observed in Ref. [3] and also discussed in Ref. [2]. The results within the stability window is thus used as initial conditions for the proposed physical system. The results in the right-hand side of Fig. 2 display the displacement of a mass calculated through numerical integration of Eq. 1. The displacement field show a very localised envelope soliton which moves around the system with a constant shape. These results suggest that cyclic and symmetric non-conservative structures can experience very strong vibration localisation when travelling waves emerge in the modulationally unstable regime.

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