Cable substructuring with delayed feedback control

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Summary. Real-time dynamic substructuring is an experimental method for testing large or complicated structures. In a substructuring test part of the system is modelled numerically and the remainder is kept as the physical substructure. The numerical-physical parts are connected via actuators and sensors and the interface is controlled by advanced algorithms to ensure that the tested structure replicates the emulated system with sufficient accuracy. The main challenge in such a test is to overcome the dynamic effects of the actuator and associated controller, that inevitably introduce delay into the substructured system which, in turn, can destabilize the experiment. This paper considers the substructuring of a taut cable clamped at both ends and, by means of the analysis of the governing coupled PDE model, investigates the stability of the system in terms of the location of the numerical-physical interface and the feedback delay. A two parameter stability diagram is presented that shows stable and unstable regions along the length of the cable for a range of delay values.

Introduction

In a real-time dynamic substructuring test part of the tested system is modelled numerically and the remainder is kept as the physical substructure [1, 2]. The interface between the two consists of actuators and sensors. Ideally, the displacements of the interface are calculated by the numerical part and imposed on the physical part with the force required to do this fed back to the numerical model. Due to the dynamic effects of the interface, this force feedback is delayed; we assume a constant, state-independent delay. In this work we consider a substructured taut cable and investigate how its free vibrations are affected by the substructuring. The interface is at an arbitrary point along its length. We investigate the stability of the substructured system in terms of the interface location and the delay. The cable is of length L, has a cross-sectional area of A and its material density is ϱ. The tension in the cable is T. This system was studied by frequency domain methods in [3].

Mathematical model of the substructured cable

In the case of a cable clamped at both ends, the first vibration frequency of the cable is $\omega_1 = (\pi/L)(\sqrt{T/\varrho A})$. The equation that describes the dynamics of the free vibration of the cable is the 1D wave equation which – when nondimensionalised with respect to the first modal frequency, ω_1, and the length of the cable – reads

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial t \partial x^2}\right). \tag{1}$$

Here, $u = u(x,t)$ is the vertical displacement along the cable, t is the nondimensional time and x is the nondimensional spacial coordinate of the position along the cable. The nondimensional structural damping is considered to be proportional to the stiffness with a coefficient γ. When the cable is split at point $x = l$ this mathematical model transforms into two coupled PDEs that are joined together by their boundary conditions at $x = l$. Note, that due to being a dimensionless parameter, l indicates what percentage of the full cable is included in the physical substructure. The two equations are given by

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u_1}{\partial x^2} + \gamma\frac{\partial^2 u_1}{\partial t \partial x^2}\right) \text{ for } \quad x \in [0,l] \quad \text{and} \quad \frac{\partial^2 u_2}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u_2}{\partial x^2} + \gamma\frac{\partial^2 u_2}{\partial t \partial x^2}\right) \text{ for } \quad x \in [l,1], \tag{2}$$

where $u_1 = u_1(x,t)$ is the vertical displacement of the physical substructure and $u_2 = u_2(x,t)$ is that of the numerical model. The boundary conditions from the clamps and the equality of the displacements at the interface are given by

$$u_1(0,t) = 0, \quad u_2(1,t) = 0 \quad \text{and} \quad u_1(l,t) = u_2(l,t), \tag{3}$$

respectively. During the substructuring experiment the vertical force $f_1 = f_1(x,t)$ measured at $x = l$ is fed back to the numerical model with gain K and (nondimensional) delay τ. From Hooke's law this force is given by $f_1(x,t) = \partial u_1/\partial x$ and, therefore, the force balance at the interface is given by

$$\frac{\partial u_2}{\partial x}(l,t) = K\,\frac{\partial u_1}{\partial x}(l,t-\tau) \tag{4}$$

In this work we consider $K = 1$ only, to model the case of no amplitude alteration of the feedback signal. This is the most basic feedback mechanism and gives an insight into how the delay affects the system in case of unity gain. The stability is studied by the characteristic equation of the system described by (2). This is obtained from the boundary conditions after the substitution of the trial solution $u_n(x,t) = \Phi_n(x)\,\mathrm{e}^{\lambda t}$, $n \in \{1,2\}$ into (2), giving

$$\frac{\lambda\pi}{\sqrt{1+\gamma\lambda}}\left(\left(1 + K\,\mathrm{e}^{-\lambda\tau}\right)\sinh\frac{\lambda\pi}{\sqrt{1+\gamma\lambda}} + \left(1 - K\,\mathrm{e}^{-\lambda\tau}\right)\sinh\frac{(2\,l-1)\,\lambda\pi}{\sqrt{1+\gamma\lambda}}\right) = 0. \tag{5}$$

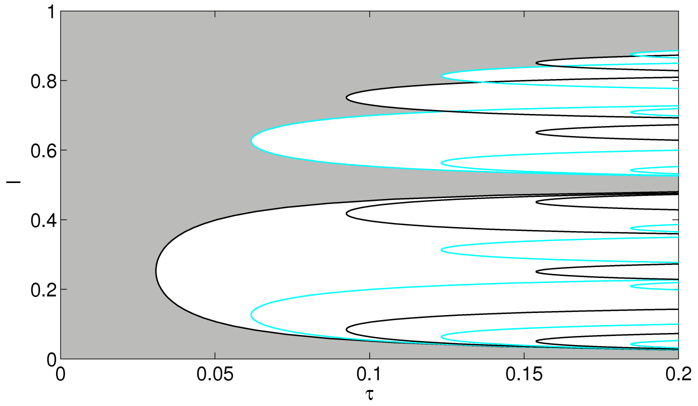

Figure 1: Two dimensional stability map of the substructured cable in the (τ, l)-plane for $\tau \in [0, 0.2]$ and $l \in [0, 1]$. The shaded area corresponds to stable equilibrium solutions of the system.

When any of the roots of (5) are purely imaginary, the system is at the limit of stability. Apart from the trivial $\lambda = 0$ solution, this may happen when $\lambda = \pm \mathrm{i}\,\omega$. Upon substituting this solution into (5) and separating the real and imaginary parts of the resulting complex equation, expressions for $l(\omega)$ and $\tau(\omega,\, l)$ may be derived which, in turn, may be used to compute the stability diagram in the $(\tau,\, l)$-plane.

Stability diagram in the $(\tau,\, l)$-plane

The stability diagram is shown in Figure 1. Note, that more than one value of ω satisfy the necessary conditions for the characteristic root to be purely imaginary and these values are associated with the modal frequencies of the freely vibrating cable. Corresponding to the multiplicity of solutions, for one value of l there are multiple values of τ. Particular values of ω form continuous curves on the $(\tau,\, l)$-plane. Black curves represent odd numbered modes whereas blue are associated with even numbered modes. Shaded regions correspond to the stable equilibria of the system. The free vibration of the emulated cable corresponds to the $\tau = 0$ case. As can be seen, in the interval of $l \in [0, 0.5]$, that is, when the physical substructure is not longer than half of the cable, the stability boundary is defined by the curve corresponding to the first vibrational mode, with the minimum tolerance to delay being at $l = 0.25$. This means that in this region if the delay is above the critical value the substructured cable experiences an unstable oscillation at a frequency close to the first natural frequency of the clamped cable. If, for given l, the delay is greater, the oscillations may exhibit a higher natural frequency. Considering the case of $l = 0.5$ the system regains stability for all values of τ. This is due to the vertical component of the interface force always being zero in this case which decouples the numerical model from the physical substructure and, therefore, the delay at the interface has no effect on the originally stable system. Moreover, $l = 0.5$ also marks the location of the mid-span zero-displacement node of the second vibration mode of the cable, the physical appearance of which has consequences on the stability in terms of the rest of the possible interface locations. Namely, a curve associated with some higher mode of the cable can only become a stability boundary if the appearance of the nodes related to the mode are not prohibited. For example, for the third mode to become the stability boundary, the physical substructure needs to be at least two-thirds of the cable length to allow the two associated nodes to appear (in practice, due to the overlap of the curves at $l > 0.5$ this threshold is more than the two-third span). The analytical closed form solutions calculated here have been validated against results from the approximate numerical finite-element based study reported in [3].

Conclusions

This paper considered a substructured taut cable whereby part of the cable is modelled numerically and the rest is kept as the physical substructure. Through the analysis of the characteristic equation of the substructured system we showed how the selected interface location may change the tolerance of the system to the delay of the actuator at the numerical-physical interface. This shows that in a substructuring experiment, careful selection of the interface is of great importance.

References

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