# Effects of an external parameter on the synchronization threshold of time-delayed Hindmarsh-Rose neurons

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*Summary*. In this work we study synchronization of two coupled Hindmarsh-Rose (H-R) neurons. By means of numerical simulations, we investigate how the synchronization threshold is affected by an external (bifurcation) parameter and time-delay in the coupling. We found that the bifurcation parameter influences the area of synchronization. Which means that, the coupling strength lower and upper bound, and the maximum time-delay admissible to achieve synchrony, change while varying the bifurcation parameter of the H-R neurons.

## Introduction

Synchronization is a persistent type of time correlated behaviour and has been observed in nature, such as, bird flocks, firing of fireflies, neuronal networks, among others [1]. Physicist and mathematicians have observed and studied these phenomena of synchronization. For instance, in the 17th century, the Dutch scientist Christiaan Huygens observed and documented the phenomenon of synchronization in two pendulum clocks hanging on a wooden beam that is supported by two chairs [2]. Later researchers have shown the importance of studies on synchronization in a vast diversity of sciences, from engineering applications such as, power grids [3]; all the way to neuroscience, which studies the underlying behaviour of synchronized neuron on the brain [7].
In this work, we consider the electrical synapse between neurons, also known as gap junction. Which is the difference between the neurons membrane potentials. There are different characteristics than influence synchronization, such as, network topology, coupling strength, external parameters, and synapses delays. The latter three studied briefly in this work.
Recent works, such as [5], and [6] have studied network of chaotic oscillators with time-delay coupling, and found conditions for synchronization. These previous theoretical works can predict the existence of a delay-dependant lower and upper bound on coupling strength that may lead to synchronization on a network of time delayed coupled oscillators. Nevertheless, how the threshold for synchronization is affected by an external parameter with time-delay coupling has not been addressed. Therefore, this work studies numerical simulations of a simple network of two coupled Hindmarsh-Rose neuron, whose behaviour is affected by an external parameter $I$, whereas the interconnection signal is delayed by a time positive constant $\tau$. Our goal is to investigate this influence on synchronization that might lead to a better understanding of the coupling mechanism among neurons in mammalian brains.

## Methods

In literature we can find various models that mimic the behaviour of a neuron, for instance, FitzHugh-Nagumo (FHN), Hindmarsh-Rose (H-R)), among others. We use the H-R model, which provides a "richer" dynamics than FHN.
We consider a network with 2 H-R neurons. Each neuron has the following dynamics:

$$\begin{aligned}\dot{z}_{i,1} &= 1 - 5y_i^2 - z_{i,1} \\ \dot{z}_{i,2} &= 0.005(4(y_i + 1.6180) - z_{i,2}) \\ \dot{y}_i &= -y_i^3 + 3y_i^2 + z_{i,1} - z_{i,2} + I + u_i,\end{aligned} \tag{1}$$

where $i = 1, 2$, $y_i$ is the output potential of the $i$-th neuron, $I$ is the bifurcation (external) parameter that regulates the behaviour of the neuron, $z_1$ and $z_2$ are internal variables, and $u_i$ is the coupling between neurons, defined as

$$u_i(t) = \sigma(y_j(t-\tau) - y_i(t-\tau)). \tag{2}$$

where positive constant $\sigma$ is the coupling strength between neurons, and $\tau$ is the time delay. The Hindmarsh-Rose model presents different modes depending of the value of the bifurcation parameter, these modes are: bursting, chaotic bursting and tonic spiking. As shown in Fig. 1.

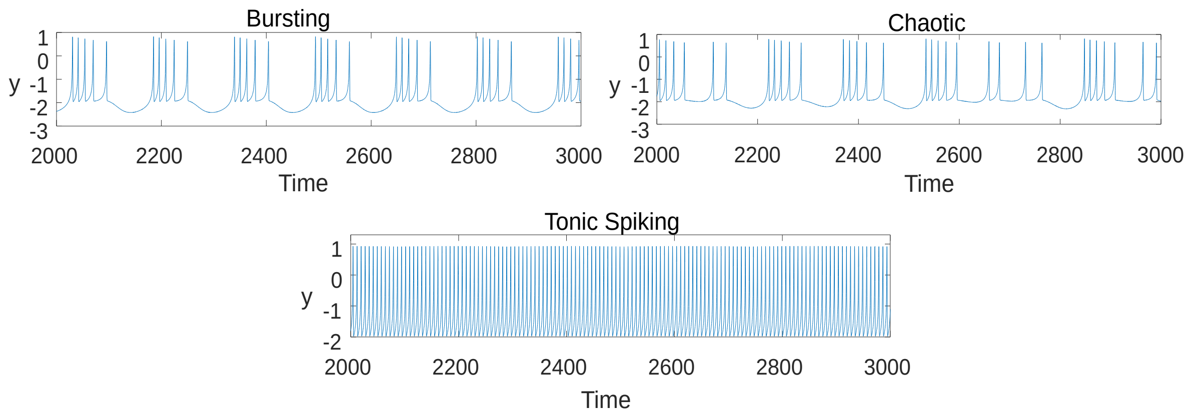
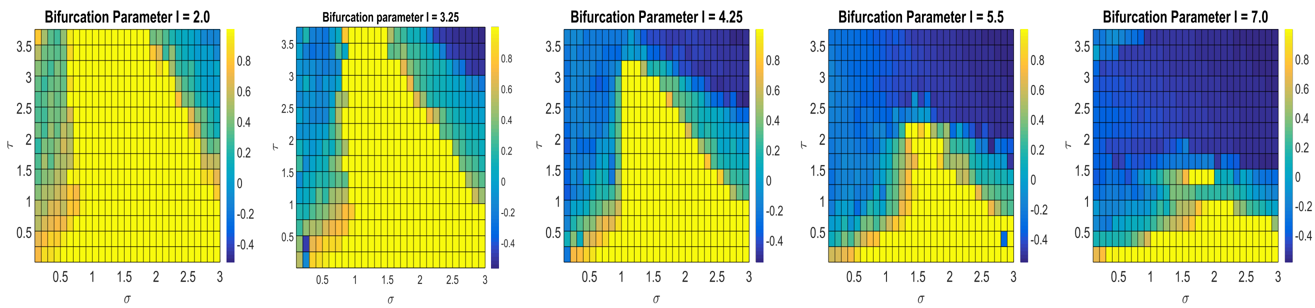



Figure 1: Membrane potential $y(t)$ changes with respect to the bifurcation parameter $I$. Bursting mode can vary from 1 to 5 spikes: $1.4 \lesssim I \lesssim 3.1$. Chaotic mode: $3.1 \lesssim I \lesssim 3.5$. Tonic Spiking: $I \gtrsim 3.5$.




(a) Bursting (b) Chaotic Bursting (c) Tonic Spiking (d) Tonic Spiking (e) Tonic Spiking

Figure 2: $(\sigma, \tau)$-Plane. Effect on the maximum time-delay coupling for different values of the bifurcation parameter, $I = [2, 3.25, 4.25, 5.5, 7.0]$.

**Numerical results**

By means of numerical simulations, we computed two time-delayed coupled H-R neurons for different values of bifurcation parameter $I$, time-delay $\tau$, and coupling strength $\sigma$. We used the concept of correlation to quantify the synchronization between neurons, visible as a colour map in Fig. 2. Where correlation equal to 1 (yellow area) represents synchrony between neurons.

In Figures 2a, b, c, d, and e, we appreciate how the synchronization area shrinks (mainly) in the vertical axis, which means that the maximum time-delay admissible to achieve synchrony decreases while the bifurcation parameter $I$ increases.

## Conclusions

The information related to the coupling mechanism of neurons and how the synchronization phenomena emerges is poorly understood. The results presented in previous works predict the existence of a $(\sigma, \tau)$-plane in the chaotic regime, however, we extended to other visible modes in the H-R model, for example bursting and tonic spiking.

In this work we computed series of extensive numerical simulations and we observed the influence of an specific parameter on the synchronization. It was visible that the minimum coupling strength and the maximum time-delay to achieve synchronization are affected by the dynamics of the individual neuron (given by the parameter $I$).

## References

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