Chaos Control in Fractional-order Systems Using Fractional Chebyshev Collocation Method

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Summary. A new method is proposed to design a fractional state feedback controller for controlling a chaotic system to a desired artificial equilibrium or periodic trajectory. For this purpose, a fractional Chebyshev collocation method is proposed to obtain Lyapunov exponents (LEs) in a nonlinear fractional order system. The dominant LE is then computed by measuring the exponential rate of the trajectory deviations initiated slightly off the attractor point. Next, a fractional state feedback controller is designed to control the chaotic system to a desired equilibrium or periodic trajectory such that the error dynamics are time invariant or time periodic, respectively. The proposed technique is implemented in a damped driven pendulum with fractional order damping and the convergence of the dominant LE is studied. Finally, the proposed technique is used to control the trajectory to a desired periodic orbit.

Introduction

Chaos control of nonlinear integer order systems has been studied extensively in the literature. One of the most popular methods is the OGY (Ott-Grebogi-Yorke) method [1] which converts the experimental time series of the system to a discrete Poincare map to stabilize unstable periodic orbits embedded in the chaotic attractor using small perturbations in the system parameters. There have been few studies on chaos control in integer order systems using time periodic control gains. A delayed state feedback controller was designed in [2] for chaos control in nonlinear periodic systems with time delay, where a symbolic approach was used to obtain the fundamental solution matrix.

In this abstract, the fractional Chebyshev collocation (FCC) method [3-5] is used to design a fractional state feedback controller to control the chaotic system to a desired artificial equilibrium or periodic trajectory in which the error dynamics have constant or periodic coefficients, respectively. Furthermore, the dominant LE of a fractional order system is obtained using a trajectory deviation technique where the FCC method is used to integrate the fractional order system. The solution of the fractional order system is discretized by N-degree Gauss-Lobatto-Chebyshev (GLC) polynomials where N is an integer. Then, the discrete orthogonality relationship for the Chebyshev polynomials is used to obtain the fractional Chebyshev differentiation matrix. The differentiation matrix is then used to convert the nonlinear fractional differential equations to a system of nonlinear algebraic equations with the collocation points as the unknowns. Numerical results are presented for a damped driven pendulum with fractional dampers. The Grünwald-Letnikov (GL) approximation technique is also employed to integrate the system when obtaining the LEs using the trajectory deviation approach and the results of the FCC and GL techniques are compared. Due to the spectral convergence of the FCC method, the results obtained by this technique are shown to be more accurate than those obtained by the GL method.

Fractional Chaos Control

Consider a nonlinear fractional system of the form

$$\ddot{x} + \mathcal{L}(x, {}_0^C\mathcal{D}_t^\eta x) + \mathcal{N}(x, {}_0^C\mathcal{D}_t^\gamma x, t) = u(t) \tag{1}$$

where $\mathcal{N}(x, {}_0^C\mathcal{D}_t^\gamma x, t)$ represents nonlinear terms, $\mathcal{L}(x, {}_0^C\mathcal{D}_t^\eta x) = a\,x + b\,{}_0^C\mathcal{D}_t^\eta x$ $(a, b \in \mathbb{R})$ denotes the linear terms, and the left-sided Caputo derivative is defined as

$${}_a^C\mathcal{D}_x^\alpha f(x) = {}_a\mathcal{J}_x^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil} f(x) \tag{2}$$

where $a \in \mathbb{R}$, $\alpha \in [0,1]$, $\lceil x \rceil$ is the ceiling function, D is the integer derivative operator, and ${}_a\mathcal{J}_x^\alpha(.)$ is the left-sided fractional integral defined as ${}_a\mathcal{J}_x^\alpha f(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^x f(\xi)(x-\xi)^{\alpha-1} d\xi$ in which $\Gamma(\cdot)$ denotes the Gamma function while $0 < \gamma < 1$ and $0 < \eta < 1$ are fractional orders. The nonlinear fractional control input $u(t)$ is split into the feedforward and feedback control terms, i.e. $u(t) = u_f(t) + u_b(t)$, and the nonlinear equation for the desired solution is $\ddot{x}_d + \mathcal{L}(x_d, {}_0^C\mathcal{D}_t^\eta x_d) + \mathcal{N}(x_d, {}_0^C\mathcal{D}_t^\gamma x_d, t) = u_f(t)$ where x_d is the desired equilibrium point or T-periodic trajectory of the form $x_d(t) = x_d(t+T)$, $e = x - x_d$ is the tracking error, and $K_p(t)$ and $K_d(t)$ are time-periodic control gains. The linearized dynamics of the tracking error are obtained by

$$\ddot{e} + \mathcal{L}(e, {}_0^C\mathcal{D}_t^\eta e) + \frac{\partial\mathcal{N}}{\partial x}\Big|_{x=x_d} e = u_b(t) \tag{3}$$

The linear fractional time-periodic feedback control law

$$u_b(t) = -K_p(t)e - K_d(t)\ {}_0^C\mathcal{D}_t^{\beta} e \quad (4)$$

is designed to drive the tracking error to zero within a certain domain of attraction. The FCC technique is used to discretize the state space form of the tracking error dynamics at the GLC collocation points $\mathbf{t}$ [3-5]. The discretized solution is obtained as $\mathbf{E} = \mathbf{M}\mathbf{E}_0$ where $\mathbf{E} = [\mathbf{E}_1^T, \mathbf{E}_2^T]^T$, $\mathbf{E}_0 = [\mathbf{E}_{10}^T, \mathbf{E}_{20}^T]^T$, and $\mathbf{M}$ is the monodromy matrix. By applying the induced norm $\|(\cdot)\|$, one can write $\|\mathbf{E}\| \le \rho_{\max}(\mathbf{M})\|\mathbf{E}_0\|$ where $\rho_{\max}(\mathbf{M})$ denotes the maximum eigenvalue of $\mathbf{M}$ known as the spectral radius. Therefore, the necessary and sufficient condition for stability of the tracking error dynamics in Eq. (3) is that all the characteristic multipliers lie inside the unit circle [3-5]. When this condition is satisfied, the system trajectory $x(t)$ asymptotically approaches the desired equilibrium point or T-periodic trajectory $x_d(t)$.

Fractional Damped Externally Driven Pendulum

Consider a damped externally driven pendulum with fractional order damper, i.e.

$$\ddot{x} + \mu\ {}_0^C\mathcal{D}_t^{\alpha} x + \omega^2 \sin x = F_0 \sin \Omega t \quad (5)$$

where the fractional damping is given by $F_\mu(t) = \mu\ {}_0^C\mathcal{D}_t^{\alpha} x(t)$ [5], $\mu = 0.5$ is the damping coefficient, and the natural and external frequencies of the system are assumed to be $\omega = 1$ rad/s and $\Omega = 2/3$ rad/s, respectively. The LEs computed for the corresponding integer order system are given in Fig. 1 using the Jacobian method (dashed black) and the trajectory deviation approach obtained by employing either the GL (solid curve) or the FCC (dotted curve) methods. The dominant LE is plotted in Fig. 2 for the fractional damped driven pendulum with $\alpha = 0.8$ and different values of the GLC collocation points. This figure shows how the results for the dominant LE converge as the number of GLC collocation points increases. According to Fig. 2, the fractional system with fractional order $\alpha = 0.8$ experiences the first pitchfork bifurcation at $F_0 \approx 1.025$ and becomes chaotic at $F_0 \approx 1.052$.

It is desired for $u(t)$ to be selected such that the chaotic system is controlled to a desired periodic trajectory of the form $x_d = \sin(t)$, which is a unit circle in the ($x - \dot{x}$) trajectory plane. The periodic control gains are selected in the form of $K_p(t) = k_{11} + k_{12}\sin(\Omega t) + k_{13}\cos(\Omega t)$, $K_d(t) = k_{21} + k_{22}\sin(\Omega t) + k_{23}\cos(\Omega t)$, where k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, and k_{23} are scalars. The fractional feedback controller is then applied to the integer and fractional order systems for the case of $F_0 = 1.085$ which corresponds to chaotic behavior in both systems in the absence of the controller as shown in Fig. 3. It is shown that the (optimal) controller $u(t) = u_f(t) + u_b(t)$ is capable of bringing the system trajectory to the desired periodic orbit.

Conclusion

New techniques to obtain the dominant Lyapunov exponent and design a linear fractional feedback controller with periodic control gains to drive the chaotic motion to a desired periodic reference trajectory were demonstrated for a fractional order system. Furthermore, the dominant LE was obtained by measuring the trajectory deviations at different time steps where fractional Chebyshev collocation and Grünwald-Letnikov techniques were used to integrate the system of equations. The proposed techniques were implemented on a damped driven pendulum with fractional order damper and the chaotic behavior of the system was studied and controlled to a periodic trajectory.

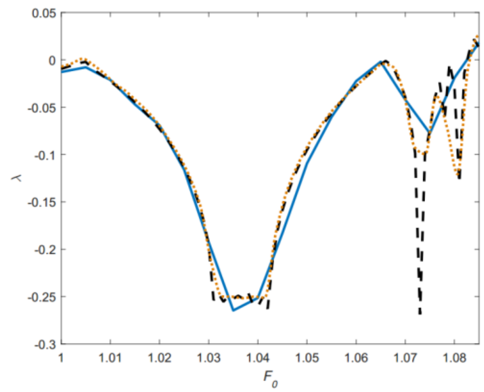

Figure 1: LE diagram obtained by the Jacobian technique (black dashed) and trajectory deviation using GL (blue solid) and FCC (red dotted) for integer order damper ($\alpha = 1.0$).

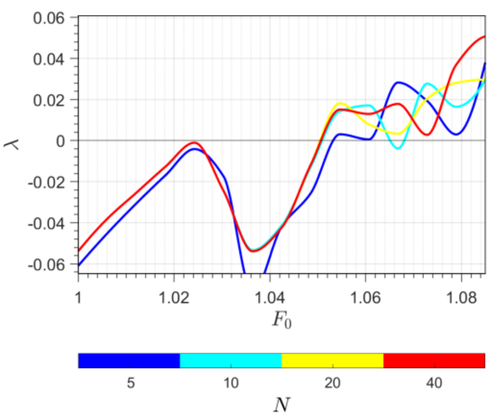

Figure 2: LE diagram by the FCC technique for fractional order damper ($\alpha = 0.8$) using different GLC collocation points N.

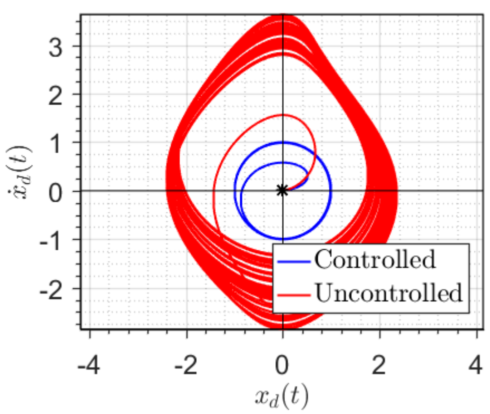

Figure 3: The response of system (6) using the feedback control in Eq. (5) to control it to a desired periodic trajectory.

References

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