Large time-periodic systems in engineering applications

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Summary. In many problems of machine dynamics the systems are modelled with finite elements, resulting in large systems of linear differential equations. These equations often should have time periodic coefficients. A typical example are the mathematical models of disk brakes, where the disk containing ventilation channels always should lead to time periodic equations. Consideration of such time periodic coefficients may be decisive for proper assessment of systems stability characteristics. The analysis is however usually carried out for constant angular positions of the disk, which avoids the time-periodic terms but does not give an accurate mathematical model. The standard application of Floquet theory in this case may become computationally prohibitive. Methods are therefore needed to deal with the problem in a FEM environment, with the aim to allow an efficient stability analysis in this context, without having to numerically compute the monodromy matrix by time integration of a very high dimensional system.

Extended abstract

Time-variant and in particular periodic mechanical systems are common in Machine Dynamics, but also in other engineering fields. The theory of linear time-periodic differential equations was developed about a hundred years ago by Floquet. In mechanical systems the linear parts of the equations of motion are characterized by the **M**, **D**, **G**, **K**, **N** matrices which may all be time-periodic (mass, damping, gyroscopic, stiffness and circulatory matrices, respectively). A typical property of parametric instability behavior is the existence of combination resonances. In Machine Dynamics, however, usually these effects are mainly relevant for extremely weakly damped systems, and they therefore are rarely observed in reality. They also tend to occur only in very narrow frequency ranges of the parametric excitation. However, if parametric excitation in the system is simultaneously present in the **K**(t) and the **N**(t) matrices, a different behavior may occur: The linear system may then be unstable for all frequencies of the parametric excitation, and not only in the neighborhood of certain discrete frequencies. In this case the term ‘parametric resonance’ is not appropriate, since there is no special resonance condition required for instability. This unusual behavior of a parametrically excited system was first observed and proved by L. Cesari [1] and mentioned several decades later by E. Mettler [2]. We refer to this situation as ‘atypical’ parametric instability.

In the past, particularly conservative and stable linear systems were studied at length; they may become unstable due to additional parametric excitation (parametric resonance). More recently, time periodic mechanical systems have also become important in the context of self-excited vibrations. In many engineering systems, self-excitation appears in the equations of motion in the form of circulatory terms (skew symmetric matrices in the coordinate proportional forces). In some engineering systems, the frequency of the parametric excitation is much lower than that of the self-excited vibrations, so that parametric resonances in the usual sense do not play a role. Even so, the periodic coefficients may be crucial for stability. An example of this type of self-excited vibrations is break squeal. It turns out that the equations of motion of a squealing brake contain all the terms which may lead to atypical parametric resonance.

In all these cases the stability can in principle of course always be studied via Floquet theory. For large periodic systems, i.e. systems with many degrees of freedom, the computational cost of calculating the monodromy matrix (needed for applying Floquet theory) may however become prohibitive [3]. The widespread use of finite element methods (FEM) frequently leads to large systems of differential equations, possibly of many thousand or even hundreds of thousands of degrees of freedom), which may contain periodic coefficients. Very early it was recognized that special techniques are needed for dealing with the problem of time-periodic systems in a FEM environment, with the aim to allow an efficient stability analysis in this environment. One of the main challenges when analyzing a large dynamic system defined via FEM is that the system matrices obtained from a finite element model are given only for a particular state even if the underlying problem is time-periodic and, therefore, the equations are different for each state within a period. Thus, the equations of motion are available only for a number of previously defined positions. This drawback leads to limitations concerning the choice of an appropriate method for stability analysis. However up to this day none of the commercial FEM codes seems to contain a package for time-periodic systems! There definitely is a need for dealing with time-periodic systems in a FEM environment.

As an example, the problem of a squealing brake will be discussed in this presentation for the case of a reasonably large number of degrees of freedom, using a condensed FEM model. Squeal is a self-excited vibration and its onset can be studied with the linearized equations of motion, i.e., as a stability problem. As the self-excited vibrations build up, they will tend towards a limit cycle, which only can be computed if the nonlinearities are taken into account. In a modern disk brake the brake disk often contains ventilation channels leading to periodic coefficients and therefore to parametric excitation. It is obvious that time-periodic coefficients may be present both in the mass as well as in the stiffness matrix. But also in the other matrices (damping matrix, gyroscopic matrix and matrix of the circulatory terms) periodic coefficients will in general be present. In industrial computations based on FEM models in the automotive industry, the influence of these time-periodic coefficients is normally disregarded and the time-periodic matrices are typically substituted by constant matrices. The stability of the system with constant coefficients is then studied by carrying out an eigenvalue analysis, and very efficient and fast algorithms are available in the FEM environment to deal with these eigenvalue problems. The eigenvalues and eigenvectors for reasonably large systems can therefore be calculated at low computational cost. The agreement in the predicted stability behavior of a brake and the laboratory tests is of course not very satisfactory. The result of this type of numerical stability analysis will of course depend on the particular angular position of the disk for which the eigenvalue analysis is carried out. If the eigenvalue problem is solved for a discrete number of angular positions, which correspond to chosen time instants ('frozen times' in the rotational motion of the disk), a whole family of eigenpairs (eigenvalues and eigenvectors) is obtained.

This set of eigenpairs computed for the different frozen times can be used to construct an approximation to the monodromy matrix. Since for each frozen time the eigenpairs can be computed in the FEM environment at relatively low cost, this may be a good numerical approach to obtain an approximated monodromy matrix for a large system, without having to integrate in time the large system of periodic equations. The construction of the monodromy matrix and the related stability analysis is thus not carried out in the FEM environment but is done a posteriori and off line. Another great advantage of this method is that it does not require the formulation of system equations of motions containing time-periodic terms, which makes it especially suitable for application to large problems defined in a FEM environment. This is one of several approaches being developed in Darmstadt for the solution of stability problems in large time-periodic systems. In the presentation, the procedure is applied to data of an industrial disk brake obtained via FEM analysis, which led to a family of complex eigenpairs. The results of the stability behavior obtained with the monodromy matrix approximated as described above are then compared to the industrial results obtained with the conventional modal analysis. It is shown that the predicted stability behavior may be quite different in these two cases. The agreement with experimental data obtained with laboratory tests for the brake has yet to be examined.

In addition to the brake modelled with FEM, also a 'minimal model' of a disk brake is considered. The bending vibrations of the disk in this model are substituted by the vibrations of a 2 dof elastically supported and rotating rigid disk, in contact with an idealized brake pad. All the parametric excitations present in the FEM model due to the asymmetries, are also present in the equations of motion of the 'minimal model'. Comparison of the stability analyses for these two different systems shows a remarkable qualitative agreement. This means that the 2 dof model really is a good 'minimal model' describing qualitatively (of course not quantitatively) the main phenomena.

The limit cycles and bifurcations both in the 'minimal model' with nonlinearities in the brake pad as well as in an academic example with atypical behavior are also discussed briefly.

References

[1] Cesari L. (1940). Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici. *Atti Accad. Italia, Mem. Cl. Sci. fis. mat. natur. 11*, 633-695

[2] Mettler E. (1968). Combination resonances in mechanical systems under harmonic excitation. *Proceedings of the 4th International Conference on Nonlinear Oscillations,* 51-70, Prague.

[3] Sinha, S.C., & Wu, D.-H. (1990). An efficient computational scheme for the analysis of periodic systems. *Journal of Sound and Vibration, Volume 151, Issue 1,* 91-117.