Lagrangian and Eulerian Coherent Structures in Complex Dynamical Systems

Mattia Serra* and George Haller*
**Institute for Mechanical Systems, ETH Zürich, Switzerland*

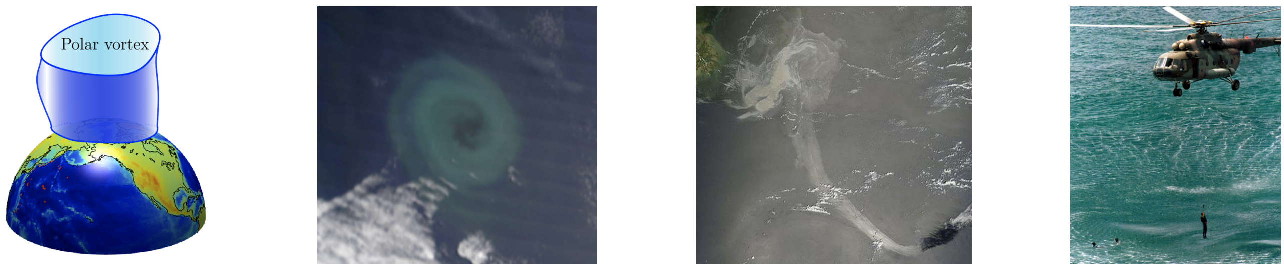
Summary. Lagrangian Coherent Structures (LCSs) and Objective Eulerian Coherent Structures (OECSs) are distinguished features in a dynamical system that create coherent trajectory patterns, and can be computed as null geodesics of appropriate Lorentzian metrics. Using the geometric structure of geodesic flows, we derive a fully-automated method for computing null geodesics of general Lorentzian metrics. We compute LCSs and OECSs in several complex dynamical systems as multi-scale Oceanic and Atmospheric Flows. Remarkably, hyperbolic OECSs reveal, instantaneously, previously undetected locations of significant deformation that exert major influence on passive and non-passive tracers over short-time scales.

BACKGROUND

Objective coherent structures (OCSs) are time-evolving surfaces that shape trajectory patterns in non-autonomous dynamical systems, such as turbulent fluid flows. Hyperbolic OCSs (generalized stable-unstable manifolds) are the main drivers of chaotic mixing in the phase space. Elliptic and parabolic OCSs (generalized KAM tori [1]), on the other hand, confine coherent patches with regular dynamics. In many applications, these structures co-exist, partitioning the phase space into regions of distinct dynamics.

Depending on the time interval over which OCSs organize nearby trajectories, they can be classified as Lagrangian Coherent Structures (LCSs) and Objective Eulerian Coherent Structures (OECSs). Specifically, LCSs [2] are influential over a finite time interval, while OECSs [5] are infinitesimally short-term limits of LCSs. LCSs are suitable for understanding and quantifying finite-time transport and mixing, intrinsically tied to a preselected time interval. OECSs, in contrast, can be computed at any time instant, and are free from any assumptions on time scales. OECSs are, therefore, relevant tools for transport forecasts [4], flow-control and real-time decision-making problems.

Figures 1a-b show examples of large scale vortices in the stratosphere and in the ocean, which owe their significance entirely to the material they carry. Specifically, the stratospheric polar vortex transports a considerable mass of cold air, and its material deformation impacts significantly earth's surface weather and seasonality phenomena. The mesoscale (100-200 km in diameter) vortex in Fig. 1b carries warm water across the Atlantic Ocean with implication for global circulation and climate change. Figures 1c-d show examples of flow-based hazards in the ocean during which reliable short-term forecasts or even now-cast are crucial to avoid catastrophic consequences.



(a) (b) (c) (d)

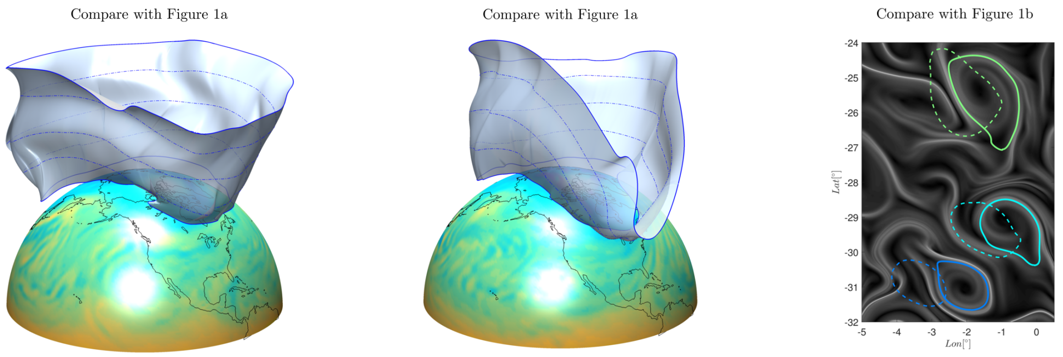
Figure 1: (a) Illustration of the stratospheric polar vortex. (b) Phytoplankton patch in the Agulhas leakage (NASA). (c) Deepwater Horizon oil spill in the Gulf of Mexico (NASA). (d) Search and Rescue operation (NATO).

Variational arguments show that OCSs can be located as null geodesics of suitably defined Lorentzian metrics [3, 5]. Their computation, however, requires a number of non-standard steps and assumptions, which in turns may preclude the detection of some OECSs or the identifications of the correct ones. We propose a fully-automated method, funded on the geometric structure of geodesic flows, that overcomes these limitations and identifies null geodesics of general Lorentzian metrics [6].

SUMMARY OF RESULTS

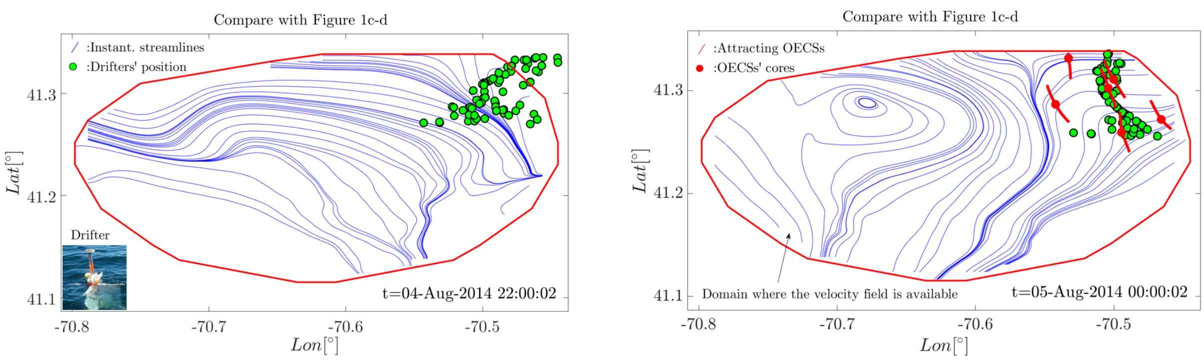
Here we apply our results to complex dynamical systems, as the ones illustrated in Fig. 1, given in the form of finite-time and finite-size datasets (cf. Fig 3).

Figures 2a-b show the stratospheric polar vortex boundary identified by ellipic LCSs [7]. The polar vortex, initially centered (Fig. 2a), deforms materially in the early January 2014 (Fig. 2b), causing an exceptional cold in the North-East coast of the US. This result was previously undetected because the polar vortex boundary is routinely identified by Eulerian diagnostics that cannot return its material evolution. Figure 2c shows coherent Lagrangian mesoscale vortices identified by elliptic LCSs in the Antarctic Ocean [6].



(a) (b) (c)

Figure 2: (a) Polar vortex boundary identified by ellipic LCSs computed from the ECMWF data on 28 Dec. 2013. (Undeformed, Fig. not to scale.) (b) Polar vortex boundary on 7 Jan. 2014. (Deformed.) (c) Lagrangian mesoscale vortices identified by elliptic LCSs computed from satellite-detected surface velocities in the Antarctic Ocean at the initial time (solid lines), together with their advected images after one month (dashed lines), on the Finite-Time Lyapunov Exponent field at the initial time.



(a) (b)

Figure 3: (a) Initial drifters' configuration close to the North-East coast of the US. Ocean surface velocity is available (measured) only within the red polygon. (b) After two hours, drifters align strikingly with Attracting OECSs.

Figures 3a shows the initial configuration of drifters (floating devices to investigate ocean currents) close to the North-East coast of the US. In the finite-size domain bounded by the red polygon, the multi-scale ocean surface velocity is measured through High-Frequency Radar (HFR), and the corresponding instantaneous streamlines are shown in blue. After two hours, drifters align strikingly with Attracting OECSs, computed using the HFR-based velocity (cf. Fig. 3b). We find this remarkable because the drifters' dynamic is different than the one of fluid particles due to wind drag and inertial effects. The initial drifters' distribution (Fig. 3a) may represent the unknown location of people landed in water or the initial distribution of an oil spill. In a situation of hazards response, therefore, attracting OECSs offers the exact locations and directions where the search and rescue effort should be redirected in a timely manner.
Figures 3 highlights two other important facts. First, Lagrangian methods are not suitable for this applications. Reasons include their dependence on a preselected time scale, while the time scales of the flow can be multiple and unknown; Lagrangian methods are inconclusive in correspondence of the initial conditions of the trajectories leaving the domain boundary. Second, frame-dependent Eulerian diagnostics routinely used to identify hyperbolic regions in the flow, such as instantaneous saddle-type stagnation points, completely miss regions of high attraction identified by Attracting OECSs.

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