Irregular dynamics of an elliptic vortex in an oscillatory nonlinear flow

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Summary. We study the dynamics of an elliptic vortex evolving in an oscillatory nonlinear flow consisting of shear and rotational components. If the external flow is stationary, the elliptic vortex can perform three types of motion: oscillation, rotation, infinite elongation. In addition, it can be motionless if positioned at a stationary point in the corresponding phase space. Certain stationary points in the corresponding phase space are hyperbolic, which makes them susceptible to external perturbations. Once perturbed the corresponding vortex dynamics features a multitude of irregular dynamical phenomena including flipping between different canonical states.

One of the most renowned vortex models is the model of an elliptic vortex subjected to linear deformation consisting of shear and rotational components [1]. In the case of stationary deformation, the model permits an elliptic vortex to perform three types of motion [1] depending on the parameters of the deformation flow and the initial alignment of the ellipse against the exterior strain. There are two periodic states involving the vortex changing its eccentricity, these are (i) rotation, and (ii) nutation. Moreover, there is one aperiodic state - infinite elongation. In this case, the vortex elongates continuously tending to be collinear with the strain axis. Moreover, for a specific initial alignment, the vortex can be stationary not performing any motion.

The model of an elliptic vortex embedded in a linear deformation field is the base model to assess the stability of elliptic vortex shapes occurring in nature. A large body of literature is devoted to the problem. Most of the papers consider spatial perturbations to the elliptic form in the case of constant linear deformation [2,3]. A prominent result of this stability analysis is that an elliptic vortex is stable to linear perturbations of its form until its geometrical shape complies with the relation $a/b \le 3$, where a and b are the major and minor semi-axes of the ellipse. An analogous ellipsoid model is used in modeling the dynamics of oceanic vortices [4,5,6].

Let us consider an inviscid, incompressible, two-dimensional flow. In this flow, an elliptic patch of constant vorticity, experiencing deformation from time-dependent strain $e(t)$ and background rotation $\gamma(t)$, is embedded. The patch conserves its elliptic form with $\varepsilon = a/b$ being the aspect ratio and φ being the angle between the ellipse's major semi-axis and the x-axis of the Cartesian coordinate frame. The governing equations are [1]

$$\dot{\varepsilon} = 2e\varepsilon\cos 2\varphi,\ \ \dot{\varphi} = \gamma + \frac{g\varepsilon}{(\varepsilon+1)^2} - e\frac{\varepsilon^2+1}{\varepsilon^2-1}\sin 2\varphi. \qquad (1)$$

The corresponding linearized homogeneous system has the form

$$\begin{aligned} \frac{d\varepsilon'}{d\tau} &= -4e(t)\varepsilon_0\varphi'\sin 2\varphi_0, \\ \frac{d\varphi'}{d\tau} &= -\varepsilon'\left[\frac{(\varepsilon_0-1)}{(\varepsilon_0+1)^3} - 4e(t)\frac{\varepsilon_0}{(\varepsilon_0{}^2-1)^2}\sin 2\varphi_0\right]. \end{aligned} \qquad (2)$$

Now, let $e(t) = \delta cos\nu t$ be the time-dependent strain. One can see that if reduced to a second order equation, the system (2) represents a Hill equation. This, in turn, signifies that there is possible the manifestation of parametric instability near the vortex stationary position. In other words, given specific values of the perturbation's parameters, the phase trajectories become unbounded near the steady-state elliptic critical point. The values of the perturbation parameters resulting in parametric instability can be readily figured out using the Floquet analysis. An analytical estimate can be derived by means of averaging techniques [7,8,9].

The linear system is a good approximation for certain values of the perturbation's parameters. For instance, when one considers the second parametric instability zone, the perturbed phase space near the steady-state elliptic critical point features no nonlinear resonances (see the Poincare section shown in fig. 1a). Therefore, the dynamics near the steady-state critical elliptic point can be derived from the linear system. When this happens, the phase trajectory that starts near the steady-state critical elliptic point moves in a spiral-like divergent trajectory (see fig. 1b)

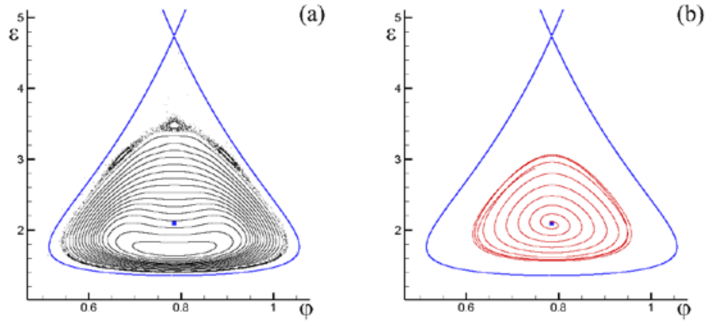

Figure 1. (a) a Poincare section illustrating that the perturbed system remains largely linear near the steady-state elliptic critical point; (b) a phase trajectory experiencing linear parametric instability that results in a divergent spiral-like trajectory.

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